Abstract

Below are all the theorems proven during the course, organized by exam questions.

One day before the exam, you will be randomly assigned a ticket. During the exam, you must present its content orally (including proofs of all statements). You are required to know all definitions that appear in your ticket (and in the proofs). Supporting materials during your presentation are not allowed.

While presenting, you may use statements from other tickets without proof. Pictures during your proofs are highly appreciated.

If you think a statement contains a typo, feel free to discuss it in a chat.

Definitions

- $d_G(v)$ – degree of the vertex v in G.
- $N_G(X)$ – all neighbours of the set X in the graph G.
- $\delta(G)$ – minimal degree in G.
- $G + ab$, is the graph G with a new edge between $a, b \in V(G)$.
- $G \cdot ab$, is the graph where the edge ab is contracted, $ab \in E(G)$.
- $c(G)$ – number of connected components.
- $o(G)$ – number of odd components.
- $\alpha(G)$ – independence number.
- $\alpha'(G)$ – maximal matching.
- $\beta(G)$ – minimal vertex cover.
- $\beta'(G)$ – minimal edge cover.
- $\chi_G(k)$ – chromatic polynomial.
- $g(G)$ – girth.
- $\kappa(G)$ – graphs connectivity.
- $\kappa_G(x, y)$ – size of the smallest set separating x and y, where $x, y \in V(G)$.
- $\kappa_G(X, Y)$ – size of the smallest set separating X and Y, where $X, Y \subset V(G)$.
- $\mathfrak{R}(G)$ – set of all separating sets of the graph G. We call the set R separating if the graph $G - R$ is disconnected.
- $\mathfrak{R}_k(G)$ – set of all k-vertex separating sets of the graph G.
- Let $\mathfrak{S} \subset \mathfrak{R}(G)$. A set $A \subset V(G)$ is a *part of the $\mathfrak{S}$-partition* if no set from $\mathfrak{S}$ separates any two vertices from A, but any other vertex of the graph G is separated from A by at least one set from $\mathfrak{S}$. The set of all parts of the partition of graph G by the separating sets $\mathfrak{S}$ will be denoted as $\text{Part}(\mathfrak{S})$.
- A vertex of a part $A \in \text{Part}(\mathfrak{S})$ is called *internal* if it does not belong to any set from $\mathfrak{S}$. The set of such vertices will be called the *interior* of part A and denoted as $\text{Int}(A)$.

  Vertices that belong to some set from $\mathfrak{S}$ are called *boundary vertices*, and their set — the *boundary* — is denoted by $\text{Bound}(A)$.

- $S, T \in \mathfrak{R}_k(G)$ – *independent* if S does not separate T and T does not separate S. Otherwise, they are *dependent*.
- $S \in \mathfrak{R}_k(G)$, H – a connected component of the graph $G - S$. We call H a *fragment*. S – boundary $\mathrm{Bound}(H)$.

Paths and Cycles

1.

Lemma 1. *Let $n \geqslant 2$, and let $a_1, \dots, a_n$ be the maximal path in the graph G, such that*

$$d_G(a_1) + d_G(a_n) \geqslant n.$$

Then the graph contains a cycle of length n.

Theorem 1 (O. Ore, 1960)**.**

a) If for any two non-adjacent vertices $u, v \in V(G)$, the condition

$$d_G(u) + d_G(v) \geqslant v(G) - 1$$

holds, then the graph G contains a Hamiltonian path.

b) If $v(G) > 2$ and for any two non-adjacent vertices $u, v \in V(G)$, the condition

$$d_G(u) + d_G(v) \geqslant v(G)$$

holds, then the graph G contains a Hamiltonian cycle.

Corollary 1 (G. A. Dirac, 1952)**.**

a) If $\delta(G) \geqslant \frac{v(G)-1}{2}$, then the graph G contains a Hamiltonian path.

b) If $\delta(G) \geqslant \frac{v(G)}{2}$, then the graph G contains a Hamiltonian cycle.

2.

Lemma 2. *Let $ab \notin E(G)$ and $d_G(a) + d_G(b) \geqslant v(G)$. Then the graph G is Hamiltonian if and only if the graph $G + ab$ is Hamiltonian.*

Corollary 2 (V. Chvátal, 1974)**.** *The graph G is Hamiltonian if and only if its closure $C(G)$ is a Hamiltonian graph.*

Lemma 3. *The closure of a graph G is uniquely determined (it does not depend on the order of edge additions).*

3.

Lemma 4. *Let the graph G be Hamiltonian. Then, for any subset $S \subset V(G)$, the inequality*

$$c(G - S) \leqslant |S|$$

holds, where $c(G - S)$ is the number of connected components in the graph $G - S$.

Theorem 2 (V. Chvátal, P. Erdős, 1972)**.** *Let $v(G) \geqslant 3$ and $\kappa(G) \geqslant \alpha(G)$. Then G is Hamiltonian.*

4.

Theorem 3 (L. Pósa, 1962). *Let* G *be a graph with* $\nu(G) = n > 2$ *satisfying the following two conditions:*

a) For any $k \in \mathbb{N}$, $k < \frac{n-1}{2}$, *the graph* G *contains fewer than* k *vertices of degree at most* k.

b) If n *is odd, then the graph* G *contains no more than* $\frac{n-1}{2}$ *vertices of degree at most* $\frac{n-1}{2}$.

Then G *is Hamiltonian.*

Theorem 4 (V. Chvátal, 1972). *Let* $0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant n-1$, *where* $n \geqslant 3$. *The following two statements are equivalent:*

a) The sequence $a_1, \ldots, a_n$ *is Hamiltonian.*

b) For every $s < \frac{n}{2}$, *if* $a_s \leqslant s$, *then* $a_{n-s} \geqslant n-s$.

5.

Theorem 5 (G. Chartrand, S. F. Kapoor, 1969). *For any connected graph* G *with* $\nu(G) \geqslant 3$ *and an edge* $e \in E(G)$, *the graph* G^3 *contains a Hamiltonian cycle that includes the edge* e.

Theorem 6 (W. T. Tutte). *Let* $k, g, n \in \mathbb{N}$, *where* $k, g \geqslant 3$, $n > k^g$, *and* kn *is even. Then there exists a regular graph* G *of degree* k *with* $g(G) = g$ *and* $\nu(G) = n$.

Matchings

6.

Lemma 5.

a) $U \subseteq V(G)$ *is an independent set iff* $V(G) \setminus U$ *is a vertex cover.*

b) $\alpha(G) + \beta(G) = \nu(G)$.

Theorem 7 (T. Gallai, 1959). *Let* G *s.t.* $\delta(G) > 0$, *then* $\alpha'(G) + \beta'(G) = \nu(G)$.

Theorem 8 (C. Berge, 1957). *A matching* M *in a graph* G *is maximum if and only if there are no* M*-augmenting paths.*

7.

Theorem 9 (P. Hall, 1935). *A bipartite graph* G *has a matching that covers all vertices of* V_1 *if and only if for any subset* $U \subset V_1$, *the following holds:*

$$|U| \leqslant |N_G(U)|.$$

Proof can be omitted.

Corollary 3. *If* $\delta(V_1) \geqslant k$ *and* $\Delta(V_2) \leqslant k$, *then there is a matching covering* V_1.

Theorem 10 (W. T. Tutte, 1947). *A graph* G *has a perfect matching if and only if for any* $S \subset V(G)$ *the following condition holds:* $o(G - S) \leqslant |S|$.

8.

Theorem 11 (Petersen, 1891). *Let* G *be a connected cubic graph with at most two bridges. Then* G *has a perfect matching.*

Example 1. *Draw a cubic graph with three edges that has no perfect matching.*

Theorem 12 (Plesnik, 1972)**.** *Let* G *be a regular with degree* k *and* $\nu(G) \equiv_2 0$*, s.t.* $\lambda(G) \geqslant k-1$. *Let* G' *be a graph obtained from* G *by removing at most* $k-1$ *edges. Then, there is a perfect matching in* G'.

Corollary 4. *Let* G *be a regular degree* k *graph with* $\nu(G) \equiv_2 0$. *Also,* $\lambda(G) \geqslant k-1$*, then for each edge* $e \in E(G)$ *there is a perfect matching containing* e.

9. A k-*factor* of a graph G is a spanning k-regular subgraph.

   Theorem 13 (J. Petersen, 1891)**.** *Every* $2k$*-regular graph has a 2-factor.*

   Corollary 5. *a) A* $2k$*-regular graph is the union of* k *of its 2-factors.*

   b) For any $r \leqslant k$*, a* $2k$*-regular graph has a* $2r$*-factor.*

10.

   Theorem 14 (C. Thomassen, 1981)**.** *Let* G *be a graph such that* $\delta(G) \geqslant k$ *and* $\Delta(G) \leqslant k+1$. *Let* $r < k$*, then there is a spanning subgraph* H *of* G *such that* $\delta(H) \geqslant r$ *and* $\Delta(H) \leqslant r+1$.

   Theorem 15 (L. Lovasz, 1970)**.** *Let* $s, t \in \mathbb{N}$*, then any graph* G *s.t.* $\Delta(G) \leqslant s+t-1$*, can be split into two graphs* H_1, H_2 *s.t.* $G = H_1 \cup H_2$ *and* $\Delta(H_1) \leqslant s,\ \Delta(H_2) \leqslant t$.

11. Let $\mathrm{Def}(G) = \nu(G) - 2\alpha'(G)$.

   Theorem 16 (C. Berge, 1958)**.** *For any graph* G *the following holds:*

$$\mathrm{Def}(G) = \max_{S \subseteq V(G)} (o(G-S) - |S|).$$

Connectivity

12. A *block* is any maximal connected subgraph of G that does not contain articulation points.

   Lemma 6. *Let* B_1 *and* B_2 *be two different blocks of the graph* G*, with* $V(B_1) \cap V(B_2) \neq \varnothing$. *Then* $V(B_1) \cap V(B_2)$ *consists of an articulation point* a *of the graph* G*, where* a *is the only articulation point separating* B_1 *from* B_2.

   Let $B(G)$ be bipartite graph, where the vertices of one part are the articulation points $a_1, \ldots, a_n$ of the graph G, and the vertices of the other part are its blocks $B_1, \ldots, B_m$. The vertices a_i and B_j are adjacent if $a_i \in V(B_j)$. The graph $B(G)$ is called the *block and articulation point tree* of the graph G.

   Lemma 7. *Let* B_1 *and* B_2 *be two different blocks of the graph* G*, and let* P *be a path between them in the graph* $B(G)$. *Then the articulation points of the graph* G *that separate* B_1 *from* B_2 *are exactly those articulation points that lie on the path* P. *Other articulation points do not even separate the union of the blocks along the path* P.

   Theorem 17. *a) The block and articulation point tree is indeed a tree, with all its leaves corresponding to blocks.*

   b) An articulation point a *separates two blocks* B_1 *and* B_2 *in the graph* G *if and only if* a *separates* B_1 *and* B_2 *in* $B(G)$.

13. We call a block B *extreme* if it corresponds to a leaf of the block and articulation point tree. The *interior* $\mathrm{Int}(B)$ of a block B is the set of all its vertices that are not articulation points in the graph G.

Theorem 18. *Let* B *be an extreme block of a connected, but not biconnected graph* G *with* $v(G) \geqslant 2$*, and let* $G' = G - \mathit{Int}(B)$*. Then the graph* G' *is connected, and the blocks of* G' *are all the blocks of* G *except for* B.

Let $U_1, \dots, U_k$ be all the connected components of the graph $G - a$, and let $G_i = G(U_i \cup \{a\})$. We decompose the graph G into the graphs $G_1, \dots, G_k$.

Lemma 8. *Let* $b \in U_i$*. Then* b *separates the vertices* $x, y \in V(G_i)$ *in* G_i *if and only if* b *separates them in* G*. All the articulation points of the graphs* $G_1, \dots, G_k$ *are exactly all the articulation points of the graph* G *except* a.

Algorithm 1. *Algorithm for Constructing the Block and Articulation Point Tree.*

14.

Theorem 19 (Menger, 1927, Goring 2000)**.** *Let* $X, Y \subset V(G)$*,* $\infty > \kappa_G(X, Y) \geqslant k$*,* $|X| \geqslant k$*,* $|Y| \geqslant k$*. Then in the graph* G*, there exist* k *disjoint* XY*-paths.*

Corollary 6. *Let vertices* $x, y \in V(G)$ *be non-adjacent,* $\kappa_G(x, y) \geqslant k$*. Then there exist* k *independent paths from* x *to* y.

Theorem 20 (Whitney, 1932)**.** *Let* G *be a* k*-connected graph. Then for any two vertices* $x, y \in V(G)$*, there exist* k *independent paths from* x *to* y.

15.

Lemma 9. *Let* $\mathfrak{S} \subset \mathfrak{R}_k(G)$, $A \in \mathrm{Part}(\mathfrak{S})$*. Then the following statements hold.*

a) *A vertex* $x \in \mathrm{Int}(A)$ *is not adjacent to any vertices in the set* $V(G) \setminus A$.
b) *If* $\mathrm{Int}(A) \neq \varnothing$*, then* $\mathrm{Bound}(A)$ *separates* $\mathrm{Int}(A)$ *from* $V(G) \setminus A$.

Lemma 10. *Let* G *be a* k*-connected graph, and let* $\mathfrak{S}, \mathfrak{T} \subset \mathfrak{R}_k(G)$.

a) *Let* $A \in \mathrm{Part}(\mathfrak{S})$*. Then* $\mathrm{Bound}(A)$ *is the set of all vertices in part* A *that are adjacent to at least one vertex in* $V(G) \setminus A$.
b) *Let* $A \in \mathrm{Part}(\mathfrak{S})$ *and* $A \in \mathrm{Part}(\mathfrak{T})$*. Then the boundary of* A *as part of* $\mathrm{Part}(\mathfrak{S})$ *coincides with the boundary of* A *as part of* $\mathrm{Part}(\mathfrak{T})$.

16.

Theorem 21. *Let* $\mathfrak{S}_1, \dots, \mathfrak{S}_n \subset \mathfrak{R}(G)$*, and let* $\mathfrak{S} = \bigcup_{i=1}^{n} \mathfrak{S}_i$*. Consider all sets of vertices of the form*

$$A = \bigcap_{i=1}^{n} A_i, \quad \text{where } A_i \in \mathrm{Part}(\mathfrak{S}_i). \tag{1}$$

Then the following statements hold:

a) *Any part* $A \in \mathrm{Part}(\mathfrak{S})$ *can be represented in the form (1).*
b) $A \in \mathrm{Part}(\mathfrak{S})$ *if and only if* A *is the maximal subset of vertices of the graph* G *representable in the form (1).*
c) *If a set of vertices* A *can be represented in the form (1) and* $A \notin \mathrm{Part}(\mathfrak{S})$*, then* A *is a subset of one of the sets in* $\mathfrak{S}$.

Lemma 11. *Let* $\mathfrak{S}, \mathfrak{T} \subset \mathfrak{R}(G)$*, and let a part* $A \in \mathrm{Part}(\mathfrak{S})$ *be such that none of the sets in* $\mathfrak{T}$ *separate it. Then* $A \in \mathrm{Part}(\mathfrak{S} \cup \mathfrak{T})$.

17.

Lemma 12. *Let* $S, T \in \mathfrak{R}_k(G)$ *and* $A \in \mathrm{Part}(S)\colon T \cap \mathrm{Int}(A) = \varnothing$. *Then* T *does not separate part* A *and, consequently,* T *does not separate set* S.

Lemma 13. *Let* $S, T \in \mathfrak{R}_k(G)$ *be such that the set* S *does not separate* T. *Then* T *and* S *are independent.*

Lemma 14. *Let* $S, T \in \mathfrak{R}_k(G)$ *be* independent, *and* $A \in \mathrm{Part}(S)$ *contain* T. *Then in* $\mathrm{Part}(T)$ *there* $\exists! B \in \mathrm{Part}(T)\colon B \supset \mathrm{Part}(S) \setminus A$ *and* $\mathrm{Part}(T) \setminus B \subset A$.

18. Let the sets $S, T \in \mathfrak{R}_k(G)$ be *dependent*, with $\mathrm{Part}(S) = \{A_1, \dots, A_m\}$, $\mathrm{Part}(T) = \{B_1, \dots, B_n\}$, $P = T \cap S$, $T_i = T \cap \mathrm{Int}(A_i)$, $S_j = S \cap \mathrm{Int}(B_j)$, and $G_{i,j} = A_i \cap B_j$.

Lemma 15.

a) *All sets* $T_1, \dots, T_m$; $S_1, \dots, S_n$ *are non-empty.*

b) $\mathrm{Part}(\{S, T\}) = \{G_{i,j}\}_{i \in [1..m], j \in [1..n]}$, *with* $\mathrm{Bound}(G_{i,j}) = P \cup T_i \cup S_j$.

Lemma 16.

a) *Let* $i \neq x$, $j \neq y$, $|\mathrm{Bound}(G_{i,j})| \geqslant k$ *and* $|\mathrm{Bound}(G_{x,y})| \geqslant k$. *Then* $|\mathrm{Bound}(G_{i,j})| = |\mathrm{Bound}(G_{x,y})| = k$, $|\mathrm{Part}(S)| = |\mathrm{Part}(T)| = 2$, $|T_i| = |S_y|$, *and* $|T_x| = |S_j|$.

b) *If all parts of* $\mathrm{Part}(\{S, T\})$ *contain at least* k *vertices, then each boundary of each part of* $\mathrm{Part}(\{S, T\})$ *has exactly* k *vertices,* $|\mathrm{Part}(S)| = |\mathrm{Part}(T)| = 2$, *and* $|T_1| = |T_2| = |S_1| = |S_2|$.

19.

Lemma 17. *Let* H *be a fragment of the graph* G, $T \in \mathfrak{R}_k(G)$, *with* $T \cap H \neq \varnothing$, *and* T *is* independent *with* $\mathrm{Bound}(H)$. *Then* $T \not\supseteq H$ *and exists a fragment* $H' \subseteq H\colon \ \mathrm{Bound}(H') = T$.

Definition 1. *A* k*-connected graph* G *is called* inseparable *if there does not exist a set* $S \in \mathfrak{R}_k(G)$ *and a fragment* H *such that* $H \subset S$.

Lemma 18. *Let* G *be a* k*-connected graph with* $\delta(G) \geqslant \frac{3k-1}{2}$. *Then* G *is inseparable.*

20.

Lemma 19. *Let* G *be an inseparable* k*-connected graph, and let* $S, T \in \mathfrak{R}_k(G)$ *be dependent. Then each of these sets divides the graph into two parts, and they can be numbered such that*

$$\mathrm{Part}(S) = \{A_1, A_2\}, \quad \mathrm{Part}(T) = \{B_1, B_2\}$$

and $|\mathrm{Bound}(G_{1,2})| = |\mathrm{Bound}(G_{2,1})| = k$. *In this numbering, we have* $|T_1| = |S_1|$ *and* $|T_2| = |S_2|$.

Theorem 22 (D. V. Karpov, A. V. Pastor, 2000). *Let* G *be an inseparable* k*-connected graph, and let* H *be a minimal fragment of* G *by inclusion. Then for any vertex* $x \in H$, *the graph* $G - x$ *remains* k*-connected.*

Corollary 7 (G. Chartrand, A. Kaugars, D. R. Lick, 1972). *Let* G *be a* k*-connected graph with* $\delta(G) \geqslant \frac{3k-1}{2}$. *Then there exists a vertex* $x \in V(G)$ *such that the graph* $G - x$ *remains* k*-connected.*

Spanning Trees

21. Denote by $\mathrm{st}(G)$ the number of spanning trees of a connected graph G.

Theorem 23 (A. Cayley, 1889)**.** *Let G be a graph where loops and multiple edges are allowed, and let $e \in E(G)$ be an edge that is not a loop. Then*

$$\mathrm{st}(G) = \mathrm{st}(G - e) + \mathrm{st}(G * e).$$

Theorem 24 (C. Cayley, 1889)**.** $\mathrm{st}(K_n) = n^{n-2}$.

22.

Theorem 25 (S. Schuster, 1983)**.** *Let a connected graph G have spanning trees with m and n leaves, where $m < n$. Then, for any natural number $k \in [m, n]$, there exists a spanning tree of G with exactly k leaves.*

23.

Theorem 26 (D. J. Kleitman, D. B. West, 1991)**.** *In a connected graph G with $\delta(G) \geqslant 3$, there exists a spanning tree with at least $\frac{v(G)}{4} + 2$ leaves.*

Coloring

24.

Lemma 20. *Let G be a connected graph, $\Delta(G) \leqslant d$, and suppose that at least one vertex of G has degree less than d. Then $\chi(G) \leqslant d$.*

Lemma 21. *If G is a biconnected but not complete graph with $\delta(G) \geqslant 3$, then there exist vertices $a, b, c \in V(G)$ such that $ab, bc \in E(G)$, $ac \notin E(G)$, and the graph $G - a - c$ is connected.*

Theorem 27 (R. L. Brooks, 1941)**.** *Let $d \geqslant 3$, and let G be a connected graph, distinct from K_{d+1}, with $\Delta(G) \leqslant d$. Then $\chi(G) \leqslant d$.*

25.

Definition 2. *Choice number* $\mathrm{ch}(G)$ *is the smallest $k \in \mathbb{N}$. Such that if one assigns list of k colors to each vertex, then there is a proper coloring, such that color of each vertex belongs to the list of that vertex.*

Example 2. *Prove that there is a graph G:* $\mathrm{ch}(G) > \chi(G)$.

Definition 3. *A graph G is called* d-choosable *if for any set of lists L satisfying the condition $\ell(v) \geqslant d_G(v)$ for each vertex $v \in V(G)$, there exists a proper coloring of the vertices of G using colors from the lists.*

Definition 4. *A vertex $v \in V(G)$ is called* normal *if $\ell(v) = d_G(v)$, and* excessive *if $\ell(v) > d_G(v)$.*

Lemma 22. *Let G be a connected graph, and let L be a d-list in which a vertex a is excessive. Then there exists a proper coloring of the vertices of G in accordance with the list L.*

Lemma 23. *Let G be a connected graph, and let L be a d-list. Suppose that there exist two adjacent vertices a and b such that the graph $G - a$ is connected and $L(a) \not\subseteq L(b)$. Then there exists a proper coloring of the vertices of G in accordance with the list L.*

26.

Definition 5. *A connected graph in which every block is either an odd cycle or a complete graph is called a* Gallai tree.

Theorem 28 (O. V. Borodin, 1977). *If a connected graph* G *is not a Gallai tree, then* G *is* d*-choosable.*

Theorem 29 (V. G. Vizing, 1976). *Let* $d \geqslant 3$*, and let* G *be a connected graph, distinct from* K_{d+1}*, with* $\Delta(G) \leqslant d$*. Then* $\mathrm{ch}(G) \leqslant d$.

27.

Definition 6. *A graph* G *is called* k-critical *if* $\chi(G) = k$*, but* $\chi(H) < k$ *for every subgraph* H *of* G.

Lemma 24. *If* G *is a* k*-critical graph, then* $\delta(G) \geqslant k - 1$.

Lemma 25. *Let* G *be a* k*-critical graph, and let* $S \subset V(G)$ *be a separating set with* $|S| < k$*. Then the graph* $G[S]$ *is not complete.*

Theorem 30 (T. Gallai, 1963). *Let* $k \geqslant 3$*, and let* G *be a* k*-critical graph. Let* V_{k-1} *be the set of all vertices of* G *with degree* $k-1$*, and let* $G_{k-1} = G[V_{k-1}]$*. Then* G_{k-1} *is a Gallai forest.*

28.

Lemma 26. *Let* G *be a non-empty graph, and let* $e = uv$ *be one of its edges. Then:*

$$\chi_{G-e}(k) = \chi_G(k) + \chi_{G \cdot uv}(k),$$

where $G \cdot uv$ *denotes the graph obtained by contracting the edge* uv.

Theorem 31. *For any loopless graph* G*, the following statements hold:*

a) *The function* $\chi_G(k) \in \mathbb{Z}[k]$ *is a monic polynomial with integer coefficients of degree* $|V(G)|$.
b) *The signs of the coefficients of* $\chi_G(k)$ *alternate (i.e., the leading coefficient is non-negative, the next coefficient is non-positive, then non-negative again, and so on).*

29.

Lemma 27. *Let* $G_1, \ldots, G_n$ *be the components of a graph* G*. Then:* $\chi_G(k) = \prod_{i=1}^{n} \chi_{G_i}(k)$.

Theorem 32. *For any graph* G*, the number* 0 *is a root of* $\chi_G(k)$ *with multiplicity equal to the number of connected components of* G.

30.

Lemma 28. *Let* G *be a connected graph with* n *blocks* $B_1, \ldots, B_n$*. Then:*

$$\chi_G(k) = \left(\frac{1}{k}\right)^{n-1} \cdot \prod_{i=1}^{n} \chi_{B_i}(k).$$

Theorem 33 (E. G. Whitehead, L.-C. Zhao, 1984). *Let* G *be a connected graph with more than one vertex. Then the number* 1 *is a root of the polynomial* $\chi_G(k)$ *with multiplicity equal to the number of blocks of the graph* G.

Planar Graphs

31.

Theorem 34 (C. Jordan, 1887.). *A closed, non-self-intersecting polygonal line P divides the points of the plane not lying on P into two parts such that the following conditions are satisfied:*

(1) Any two points from the same part can be connected by a polygonal line that does not intersect P;

(2) Any polygonal line connecting two points from different parts intersects P.

Proof can be omitted.

Define planar graph, plane graph, face, boundary of a face, walk of a boundary, $B(d)$, $Z(d)$.

Lemma 29. *A graph is planar if and only if it can be drawn on the sphere without edge crossings in its interior.*

Lemma 30.

$$\sum_{d \in F(G)} b(d) = 2e(G).$$

Lemma 31. *(a) Any two points on the boundary of a face d can be connected by a polygonal line lying entirely within d.*

(b) If two points A and B in the drawing of a graph G can be connected by a polygonal line L that does not intersect the drawing of G, then A and B lie on the boundary of some face.

Lemma 32. *Let ab_1 and ab_2 be two adjacent edges at vertex a. Then the edges ab_1 and ab_2 lie on the boundary of some face.*

32.

Lemma 33. *For a plane graph G, the following statements hold:*

(a) If $d \in F(G)$ and $B(d)$ is disconnected, then the different connected components of the graph $B(d)$ lie in different connected components of the graph G.

(b) The graph G is disconnected if and only if it has a face with a disconnected boundary.

Lemma 34. *(a) The internal edges of the faces of a plane graph G are precisely all the bridges of the graph G.*

(b) Let d be a face of an edge-2-connected graph G. Then $B(d)$ is a cycle (not necessarily simple).

Lemma 35. *If two distinct faces f and f' of a plane graph G have identical boundaries, then G is a simple cycle.*

33.

Lemma 36. *Let G be a plane graph.*

(a) If a face d and its boundary vertex a are such that B_1 and B_2 are different components of the graph $B(d) - a$, then B_1 and B_2 lie in different components of the graph $G - a$. In particular, a is an articulation point of the graph G.

(b) A graph G without self-loops is vertex-2-connected if and only if the boundaries of its faces are simple cycles.

Definition 7.
- *A cycle C of a graph G is* non-separating *if the graph $G - V(C)$ is connected.*
- *A cycle C is* induced *if it has no chords (i.e., it is an induced subgraph on its vertex set).*

Lemma 37. *Let G be a 3-connected plane graph. Then the set of boundaries of its faces is exactly the set of its non-separating induced cycles.*

Definition 8. *Let G and G' be two plane graphs, and let $\varphi : V(G) \to V(G')$ be a bijection satisfying the following conditions:*

(a) $xy \in E(G) \iff \varphi(x)\varphi(y) \in E(G')$;

(b) $U \subseteq V(G)$ *is the set of boundary vertices of some face of* G *if and only if* $\varphi(U) = \{\varphi(x) : x \in U\}$ *is the set of boundary vertices of some face of* G'.

Then φ is called an isomorphism of plane graphs G *and* G'*, and the plane graphs themselves are called* isomorphic.

Theorem 35 (H. Whitney, 1933). *Any two plane drawings of a 3-connected graph G are isomorphic as plane graphs.*

34.

Theorem 36 (L. Euler, 1752). *Let G be a plane graph with v vertices, e edges, and f faces, having k connected components. Then*

$$v - e + f = 1 + k.$$

No proof is needed.

Corollary 8. *Let G be a planar graph without loops or multiple edges, and $v \geqslant 3$. Then the following statements hold:*

(a) $e \leqslant 3v - 6$.

(b) If G is bipartite, then $e \leqslant 2v - 4$.

Corollary 9. *If G is a planar graph without loops or multiple edges, then* $\delta(G) \leqslant 5$.

Corollary 10. *The graphs K_5 and $K_{3,3}$ are non-planar.*

Lemma 38. *Let $x, y \in V(G)$, $xy \in E(G)$. Then the following assertions hold:*

1) If $G \cdot xy \supseteq K_{3,3}$, *then* $G \supseteq K_{3,3}$.

2) If $G \cdot xy \supseteq K_5$, *then* $G \supseteq K_5$ *or* $G \supseteq K_{3,3}$.

35.

Theorem 37 (K. Kuratowski, 1930). *A graph G (possibly with multiple edges and loops) is non-planar if and only if G contains a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Constructible Graphs

36.

Definition 9 (Hajós construction, 1961). *Let $q \in \mathbb{N}$. The class of q-constructible graphs $\mathcal{C}_q$ consists of graphs that can be obtained from K_q by any sequence of the following two operations:*

- *If* $G \in \mathcal{C}_q$, $x, y \in V(G)$, $xy \notin E(G)$, *then* $G\#xy \in \mathcal{C}_q$.
- *Let* $G_1, G_2 \in \mathcal{C}_q$, $V(G_1) \cap V(G_2) = \{x\}$, $xy_1 \in E(G_1)$, $xy_2 \in E(G_2)$. *Then*

$$(G_1 - xy_1) \cup (G_2 - xy_2) + y_1y_2 \in \mathcal{C}_q.$$

Base: $K_q \in \mathcal{C}_q$.

Lemma 39 (Hajós, 1961)**.** *Let* $q \in \mathbb{N}$ *and a graph* G *contains subgraph from* $\mathcal{C}_q$*, then* $\chi(G) \geqslant q$.

37.

Definition 10 (Ore, 1967)**.** *(a) Define the operation of* graph merging*. Let* G_1, G_2 *be graphs with* $V(G_1) \cap V(G_2) = \varnothing$*, and let* $W_1 \subset V(G_1)$*,* $W_2 \subset V(G_2)$ *be such that* $|W_1| = |W_2|$ *and* $\mu : W_1 \to W_2$ *is a bijection. Suppose* $x_1y_1 \in E(G_1)$*,* $x_2y_2 \in E(G_2)$*, where* $x_1 \in W_1$*,* $\mu(x_1) = x_2$*, and* $\mu(y_1) \neq y_2$ *(possibly,* $y_1 \notin W_1$*).*

The merging *of graphs* G_1 *and* G_2 *is the graph* $G_1 \#_{\mu, x_1y_1, x_2y_2} G_2$*, obtained from*

$$(G_1 - x_1y_1) \cup (G_2 - x_2y_2) + y_1y_2$$

by merging the pairs of vertices $v, \mu(v)$ *for all* $v \in W_1$*.*

(b) Let $q \in \mathbb{N}$*. Define* $\mathcal{C}'_q$ *as the class of all graphs that can be obtained from* K_q *by any sequence of graph merging operations.*

Lemma 40 (Ore, 1967)**.** *For any* $q \geqslant 3$*,* $\mathcal{C}'_q \subset \mathcal{C}_q$*.*

Lemma 41 (A. Urquhart, 1997.)**.** *Let* $q \in \mathbb{N}$*. Then any graph* G *with* $\chi(G) \geqslant q$ *can be obtained using graph merging operations from graphs containing cliques of size at least* q*.*

Hypergraph Coloring

38.

Definition of a hypergraph.

Definition 11. *A coloring of the vertices of a hypergraph* $\mathcal{H}$ *is called* proper *if every hyperedge contains at least two vertices of different colors.*

Definition 12. *An* image *of a hypergraph* $\mathcal{H}$ *is any graph* G *(possibly with multiple edges) such that* $V(G) = V(\mathcal{H})$ *and there exists a bijection* $\varphi : E(G) \to E(\mathcal{H})$ *such that* $e \subseteq \varphi(e)$ *for every edge* $e \in E(G)$*. We call* φ *the bijection of the image* G*.*

Definition 13. *Let* $r \geqslant 3$*, and let* G *be an image of the hypergraph* $\mathcal{H}$*. Consider a sequence of vertices* $a_0b_0a_1b_1 \dots a_n$ *of the hypergraph* $\mathcal{H}$ *satisfying the following conditions:*

(1) For each $i \in [0, n-1]$*, the vertices* a_i*,* b_i*, and* a_{i+1} *are distinct, and there exists a hyperedge* $e_i \in E(\mathcal{H})$ *such that* $a_i, b_i, a_{i+1} \in e_i$*.*

(2) All hyperedges $e_0, \dots, e_{n-1}$ *are distinct, and* $a_0b_0, \dots, a_{n-1}b_{n-1} \in E(G)$*, with* $\varphi(a_ib_i) = e_i$*.*

Then $a_0b_0a_1b_1 \dots a_n$ *is called an* alternating chain *from* a_0 *to* a_n*. The number* n *is called the* length *of this alternating chain. We say that the alternating chain passes through the vertices* $a_0, b_0, \dots, a_n$ *and the edges* $a_0b_0, \dots, a_{n-1}b_{n-1}$*.*

Lemma 42. *Let* $\mathcal{H}$ *be a hypergraph, where every hyperedge contains at least* r *vertices, with* $\Delta(\mathcal{H}) = \Delta$ *and* $k = \lceil \frac{2\Delta}{r} \rceil$*. Then there exists an image* G *of the hypergraph* $\mathcal{H}$ *such that* $\Delta(G) \leqslant k$*.*

39.

Theorem 38 (H.V. Gravin, D.V. Karpov, 2011)**.** *Let* $\mathcal{H}$ *be a hypergraph where each hyperedge contains at least* r *vertices,* $\Delta(\mathcal{H}) = \Delta$*, and* $k = \lceil \frac{2\Delta}{r} \rceil$*.*

(a) The vertices of $\mathcal{H}$ *can be properly colored with* $k + 1$ *colors.*

(b) If $r \geqslant 3$ *and* $k \geqslant 3$*, then the vertices of* $\mathcal{H}$ *can be properly colored with* k *colors.*