Advanced Graph Theory, Homework 1: Cycles and Matchings

1. Let G be a connected graph such that $\delta(G) \geq 2$. Prove that G has a simple path or cycle of length at least $\min\{2\delta(G), v(G)\}$.

2. $v(G) = n$, $e(G) = 5n$ and $g(G) \geq 5$. Prove that there exists at least 5 cycles that are pairwise vertex-disjoint.

3. Let G be an Eulerian graph. Let $L(G)$ be a line graph of G (i.e. vertices of $L(G)$ are edges of G and two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are incident). Prove that $L(G)$ is Hamiltonian.

4. Prove that if all simple cycles in graph G have length that is divisible by some $n \geq 3$, then $\delta(G) \leq 2$.

5. Does there exist a graph G such that it is regular with degree 2023, and it has no spanning regular subgraph?

6. Let G be a graph with $v(G) = 2021$, also G is Hamiltonian and $g(G) \geq 4$. Prove that $\exists v \in V(G)\colon \deg(v) \leq 808$.

7. Let G be a graph such that $\delta(G) \geq n$ and $g(G) \geq 5$. Prove that $v(G) \geq n^2$.

8. Given a tree T. In one turn, you can take a leaf l and remove an edge incident to l, but after that you must add an edge between l and some vertex of T. What is the minimum number of operations needed to transform one tree T_1 into tree T_2 ($v(T_1) = v(T_2) = n$). Vertices are not labeled, they are all equal.

9. Let $G = A \sqcup B \sqcup C$. And A, B, C are independent sets (tripartite graph). Also, $|A| = |B| = |C| = n$ and each vertex has at least $\frac{3n}{4}$ vertices in each of two other parts. Prove that G can be split into n triangles.

10. Edges of complete graph with n vertices (K_n) are colored. Each color is used no more than $n-2$ times. Prove that there exist three vertices u, v, w such that all edges between them are of different colors.

11. $\Delta(G) \leq 2000$ (maximum degree of G). Prove that we can color halves of edges (color for one side and color for the other side) in colors from the set $\{1, 2, 3, \ldots, 2000\}$ such that halves of a single edge are differ by exactly 1. And that all halves incident to any vertex are colored differently.