Advanced Graph Theory, Homework 2: Common Problems

1. Given bipartite graph G with n vertices in each part, such that $\delta(G) \geq \frac{n}{2}$, prove that there is a perfect matching in G.

2. Prove that every graph with average degree at least $2k$ has an induced subgraph with minimum degree at least $k+1$.

3. Prove that each graph G such that $\Delta(G) = k$ can be supplemented to a graph H, such that $V(G) \subseteq V(H)$, $E(G) \subseteq E(H)$, and $\delta(H) = \Delta(H) = k$.

4. Sequence $d_1, \ldots, d_n$ is *graphical* if there is a graph G with such degree sequence. Prove that sequence $(s, t_1, \ldots, t_s, d_1, \ldots, d_n)$ (ordered descending) is graphical iff $(t_1 - 1, \ldots, t_s - 1, d_1, \ldots, d_n)$ is graphical.

5. Let $n \geq 5$. The edges of the complete graph K_n are colored black and white.

   (a) Prove that the vertices of K_n can be divided into two groups V_1 and V_2 such that there exists a Hamiltonian path in $G(V_1)$ consisting of white edges, and a Hamiltonian path in $G(V_2)$ consisting of black edges.

   (b) Suppose n is odd. Prove that the vertices of K_n can be divided into two groups V_1 and V_2 such that there exists a Hamiltonian path in $G(V_1)$ consisting of white edges, and a Hamiltonian cycle in $G(V_2)$ consisting of black edges. It is allowed for one of the groups to contain exactly one vertex.

   (c) For which even $n \geq 5$ is the statement of part (b) true?