Advanced Graph Theory, Homework 3: Basics of Connectivity

For a connected graph G, we say that diameter $\text{diam}(G) = \max\limits_{u,v \in V(G)} d(u,v)$, where $d(u,v)$ is the distance between u and v in G.

1. Prove that if $\text{diam}(G) \geq 4$, then $\text{diam}(\overline{G}) \leq 2$.

2. Prove that any tree T has at most one perfect matching. Also prove that a tree T has a perfect matching iff $o(T - v) = 1$ for any $v \in V(T)$.

3. Let $G = (U, V)$ be a connected bipartite graph, prove that if in U all degrees are different, then there is a perfect matching in G.

4. Let $G = (U, V)$ be a bipartite graph and let $A \subseteq U, B \subseteq V$ be some sets of vertices. Assume there is M – matching covering A and M' – matching covering B, prove that there is a matching which covers $A \cup B$.

5. Is any tree of blocks and articulation points a tree of some graph?

6. A connected graph has an Eulerian cycle. Prove that each of its blocks also has an Eulerian cycle.

7. A connected graph G is called a *cactus* if every edge of G belongs to exactly one simple cycle.
   (a) Prove that all blocks of a cactus are simple cycles.
   (b) Prove that if all cycles in a graph without bridges are odd, then this graph is a cactus.
   (c) What is the maximum number of edges that a graph with all cycles odd can have (express your answer in terms of the number of vertices)?

8. Let G be a 2-connected graph with n vertices, and let $v_1, v_2 \in V(G)$ be two vertices. Let n_1 and n_2 be natural numbers such that $n_1 + n_2 = n$. Then the vertex set of graph G can be partitioned into two connected subsets $V_1 \ni v_1$ and $V_2 \ni v_2$ such that $|V_1| = n_1$ and $|V_2| = n_2$. (A vertex set is called connected if the induced subgraph on it is connected.)