Advanced Graph Theory, Homework 4: Connectivity

1. **(a)** Prove that in any biconnected graph, there is a cycle containing any two given edges.

   (b) Is it true that in any triconnected graph, there is a cycle containing any three given edges with no common endpoints?

2. Let G be a biconnected but non-bipartite graph. Prove that for any vertex $a \in V(G)$, there is an odd cycle in G passing through a.

3. It is known that among any $k+1$ vertices of a graph G, there is at least one edge. Prove that the vertices of the graph G can be covered by k disjoint simple paths.

4. Let G be a biconnected graph with $v(G) \geq 4$ and $e \in E(G)$. Prove that at least one of the graphs $G - e$ and $G \cdot e$ is biconnected.

5. The graph G is connected. We call a *separator* a minimal set of vertices whose removal disconnects the graph. Let S and R be separators of the graph G. It is known that S does not separate R. Prove that R does not separate S (it is possible that $|R| \neq |S|$).

6. Prove that any graph with a minimum degree of $2k$ contains a $(k+1)$-edge-connected subgraph. (A graph is called n-edge-connected if it remains connected when fewer than n edges are removed.)

7. The edges of a complete graph on

   (a) 4000;

   (b) 1000

   vertices are colored in three colors. Prove that this graph contains a monochromatic simple cycle of odd length at least 41.

8. (5, hw2) Let $n \geq 5$. The edges of the complete graph K_n are colored black and white.

   (a) Prove that the vertices of K_n can be divided into two groups V_1 and V_2 such that there exists a Hamiltonian path in $G(V_1)$ consisting of white edges, and a Hamiltonian path in $G(V_2)$ consisting of black edges.

   (b) Suppose n is odd. Prove that the vertices of K_n can be divided into two groups V_1 and V_2 such that there exists a Hamiltonian path in $G(V_1)$ consisting of white edges, and a Hamiltonian cycle in $G(V_2)$ consisting of black edges. It is allowed for one of the groups to contain exactly one vertex.

   (c) For which even $n \geq 5$ is the statement of part (b) true?

   Hint: Prove that there is a Hamiltonian path, such that at first it uses only white edges, and then only black edges.