Advanced Graph Theory, Colourings

1. Graph G: $\Delta(G) \leq d$, prove that its vertices can be coloured into $d^2 + 1$ colours, such that the distance between two vertices of the same colour is at least 3.

2. The plane is divided into regions by a finite number of straight lines. Prove that the regions can be coloured into 2 colours, such that the regions of the same colour do not share a common boundary.

3. Degree sequence of a graph G is $d_1 \geq \ldots \geq d_n$. Prove that

$$\chi(G) \leq \max_i \left(\min\{i, d_i + 1\}\right).$$

4. For any three cycles of odd length, at least two have a common vertex. Prove that $\chi(G) \leq 8$.

5. $\delta(G) = \Delta(G) = 3,\ v(G) = 718$. Prove that one can delete 240 vertices such that there would be no cycles of length 5 in the graph.

6. $\delta(G) = \Delta(G) = 3,\ \chi(G) \leq 3,\ a \in V(G)$. Prove that there is a proper colouring of G into 3 colours, such that the neighbours of a are not all the same colour.

7. Given 100 vertices (numbered). How many valid graphs are there, where all vertices have odd degree and the number of edges is odd?

8. Given a tournament (complete directed graph) G such that $v(G) = 1000$. Is there always a sequence of vertices $a_1, \ldots, a_{21}$ such that for any $i < j$ there is an edge $a_j \to a_i$?