Advanced Graph Theory, Homework 6: Colorings+

1. In a graph G, any two simple odd cycles have no common edges. Prove that the graph G could be colored into 2 colors, such that each vertex has at most one neighbour of the same color.

2. Let G be a Gallai tree (each block of the graph G is either an odd cycle or a complete graph). Prove that G is not d-colorable.

3. A vertex coloring is called t-periodic if the endpoints of any simple path of length t are colored the same.

   (a) Suppose the vertex coloring ρ of a connected graph G is 2-periodic. Prove that ρ colors the vertices with no more than two colors.

   (b) Let $t \in \mathbb{N}$, $t \geq 3$, and for a connected graph G, the inequality $e(G) \geq t \cdot v(G)$ holds. Assume the vertex coloring ρ is t-periodic. Prove that ρ colors the vertices with no more than two colors.

4. Prove that every k-critical graph is $(k-1)$-edge-connected.

5. The edges of a complete graph with $n \geq 3$ vertices are colored red and blue. It is allowed to change the colors of the edges in any triangle to the opposite colors. What is the maximum number of monochromatic edges that can be guaranteed to obtain with such operations?

6. Prove that for every $k \in \mathbb{N}$, there exists a bipartite graph whose list chromatic number ($\mathrm{ch}(G)$) is at least k.

7. A graph is called *chordal* if every cycle of length greater than 3 contains a chord (i.e., is not an induced subgraph).

   (a) Prove that for any incomplete chordal graph G, there exists a partition $V(G) = V_1 \cup V_2 \cup K$ of its vertices into disjoint non-empty subsets such that K is a clique in G, and the sets V_1 and V_2 are not adjacent.

   (b) A graph G is called *perfect* if for any of its induced subgraph H, the condition

   $$\chi(H) = \omega(H)$$

   holds.

   Deduce from part (a) that every chordal graph is perfect.