Advanced Graph Theory, Homework 7: Planar Graphs

1. Provide an example of a 2-connected planar graph with an arbitrarily large number of vertices, which has two non-isomorphic planar embeddings.

2. Prove that any convex polyhedron has either a vertex of degree 3 or a face that is a triangle. (A convex polyhedron is a 3-connected graph without loops or multiple edges, which can be embedded on a sphere without crossings.)

3. All faces of a polyhedron are triangles. Each face is colored black or white in such a way that the number of edges shared by two faces of the same color is minimized. Let a and b be the numbers of black and white faces, respectively. Prove that $a \leq 1.5b$.

4. Let $n > k \geq 2$ be natural numbers. In a graph G with n vertices, among any k vertices, there is a vertex adjacent to all the other $k-1$. For which n and k can it be stated that G has a vertex adjacent to all others?

5. Prove that the faces of a planar graph can be properly 2-colored if and only if the degrees of all vertices are even.