Lecture 10, Planar Graphs

11.12.2024

Content

1. Definitions
2. Walk of a Face
3. Connectivity
4. Isomorphism

Definition

A graph is called *planar* if it can be drawn on the plane such that its edges do not intersect at interior points. The vertices are represented as points, and the edges are represented as polygonal lines.

Definition

A *plane graph* (or *planar drawing*) is a specific drawing of a planar graph on the plane without intersections or self-intersections of edges.

- Thus, a planar graph can correspond to different plane graphs.

Face of a Plane Graph

- A drawing of a plane graph divides the plane into regions called *faces*.
- Denote the set of all faces of the plane graph G by $F(G)$, and let $f(G) = |F(G)|$.
- Any two points in the same face of G can be connected by a polygonal line that does not intersect the drawing of G.
- Any polygonal line connecting two points in different faces must intersect the drawing of G.

Picture!

The Plane and the Sphere

- The plane and the sphere can be transformed into one another via *stereographic projection*.
- Place a sphere on the plane, calling the point of contact the *south pole*, and the opposite point the *north pole* N. Every point $A \neq N$ on the sphere corresponds to the intersection of the plane with the ray NA.

Lemma

A graph is planar if and only if it can be drawn on the sphere without edge crossings in its interior.

Proof.

To transform a drawing of a graph from the sphere to the plane, one simply needs to choose the north pole such that it does not coincide with any vertex of the graph and does not lie on any edge. □

- A planar graph drawing on the plane is bounded (it can be enclosed in a circle).
- Therefore, in a planar drawing of a graph, there is exactly one unbounded *outer face*, which visually stands out from the other faces.

- A planar graph drawing on the plane is bounded (it can be enclosed in a circle).
- Therefore, in a planar drawing of a graph, there is exactly one unbounded *outer face*, which visually stands out from the other faces.
- The face of a spherical drawing that contains the north pole corresponds, under stereographic projection, to the outer face of the planar drawing.
- Thus, by moving the north pole to different faces of the spherical drawing, any face can be made the outer face in the planar drawing of the graph.
- This emphasizes that the outer face is not inherently different from the other faces.

- Consider an edge e of a plane graph G. Either different faces lie on opposite sides of e (in which case e is a *boundary edge* of these two faces), or the same face lies on both sides of e (in which case e is called an *internal edge* of that face). Denote by E_d the set of all boundary and internal edges of a face d.
- The boundary and internal edges of a face d are exactly the edges that can be reached from an interior point of d by a polygonal line that does not intersect the drawing of the graph.
- *Boundary vertices* of a face d are the vertices that can be reached by a polygonal line from interior points of d, without intersecting the drawing of the graph G. Denote the set of these vertices by V_d. The endpoints of edges in E_d are precisely the vertices in V_d.
- The *boundary* of a face d is the subgraph $B(d)$ of G with vertex set V_d and edge set E_d.
- The *size* of the boundary of a face d, denoted $b(d)$, is defined as the number of boundary edges of the face plus twice the number of internal edges.
- $$\sum_{d \in F(G)} b(d) = ?$$

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- The *size* of the boundary of a face d, denoted $b(d)$, is defined as the number of boundary edges of the face plus twice the number of internal edges.
- $$\sum_{d \in F(G)} b(d) = 2e(G).$$

Lemma

1. *Any two points on the boundary of a face d can be connected by a polygonal line lying entirely within d.*
2. *If two points A and B in the drawing of a graph G can be connected by a polygonal line L that does not intersect the drawing of G, then A and B lie on the boundary of some face.*

Proof.

On the whiteboard. □

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Definition

Consider any vertex a of a plane graph G, and order the edges incident to a in clockwise order. Two edges whose order is consecutive in this arrangement are called *adjacent at vertex a*.

Lemma

Let ab_1 and ab_2 be two adjacent edges at vertex a. Then the edges ab_1 and ab_2 lie on the boundary of some face.

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Boundary Walk of a Face

- Let G be a plane graph, $d \in F(G)$, and $x_1x_2 \in E_d$.
- Traverse the edge x_1x_2 from x_1 to x_2. W.l.o.g. the face d is located to the right as we proceed along the edge.
- At vertex x_2, turn right and follow the next adjacent edge x_2x_3. (If $d_G(x_2) = 1$, then $x_3 = x_1$, which causes no issue.) Clearly, $x_2x_3 \in E_d$.
- Traverse the edge x_2x_3 from x_2 to x_3, with the face d still to the right. Continue this process. Eventually, we will return to the edge x_1x_2 (or even earlier, back to vertex x_1).
- This process forms a closed cyclic route.

Picture!

- Let a cyclic route $Z = x_1 x_2 \dots x_k$ be formed. Consider a vertex x_i. By construction, Z moves around x_i—say, counterclockwise. Suppose we leave x_i along edge $x_i x_{i+1}$, and later return to this vertex via edge $x_j x_{j-1}$ (in this case, $x_i = x_j$).
- The sector between the outgoing edges $x_i x_{i+1}$ and $x_j x_{j-1}$ from vertex $x_i = x_j$ does not belong to face d.
- Thus, Z traverses all edges from E_d incident to vertex x_i. Since this is true for every vertex in Z, the route traverses exactly all edges of one component of the graph $B(d)$.
- Denote by $Z(U)$ such a route for a component U, and by $Z(d)$ the union of all constructed routes for all components of $B(d)$.
- If the route $Z(d)$ traverses an edge e twice, then, evidently, it does so in opposite directions. This means that face d lies on both sides of e, i.e., e is an internal edge of d.
- Suppose e is an internal edge of face d (see edge $x_2 x_3 = x_6 x_7$ in the figure). Then, when traversing e in either direction, face d will always lie to the right. Thus, the route $Z(d)$ traverses e twice—in both directions.

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Lemma

For a plane graph G, the following statements hold:

1. *If $d \in F(G)$ and $B(d)$ is disconnected, then the different connected components of the graph $B(d)$ lie in different connected components of the graph G.*
2. *The graph G is disconnected if and only if it has a face with a disconnected boundary.*

Proof.

On the whiteboard. □

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Lemma

1. *The internal edges of the faces of a plane graph G are precisely all the bridges of the graph G.*
2. *Let d be a face of an edge-2-connected graph G. Then $B(d)$ is a cycle (not necessarily simple).*

Proof.

On the whiteboard. □

Can a graph G contain two faces with same boundaries?

Lemma

If two distinct faces f and f' of a plane graph G have identical boundaries, then G is a simple cycle.

Proof.

On the whiteboard. □

Lemma

If two distinct faces f and f' of a plane graph G have identical boundaries, then G is a simple cycle.

Lemma

Let G be a plane graph.

1. *If a face d and its boundary vertex a are such that B_1 and B_2 are different components of the graph $B(\mathrm{d}) - a$, then B_1 and B_2 lie in different components of the graph $G - a$. In particular, a is an articulation point of the graph G.*
2. *A graph G without self-loops is vertex-2-connected if and only if the boundaries of its faces are simple cycles.*

Proof.

On the whole whiteboard. □

Definition

- A cycle C of a graph G is *non-separating* if the graph $G - V(C)$ is connected.
- A cycle C is *induced* if it has no chords (i.e., it is an induced subgraph on its vertex set).

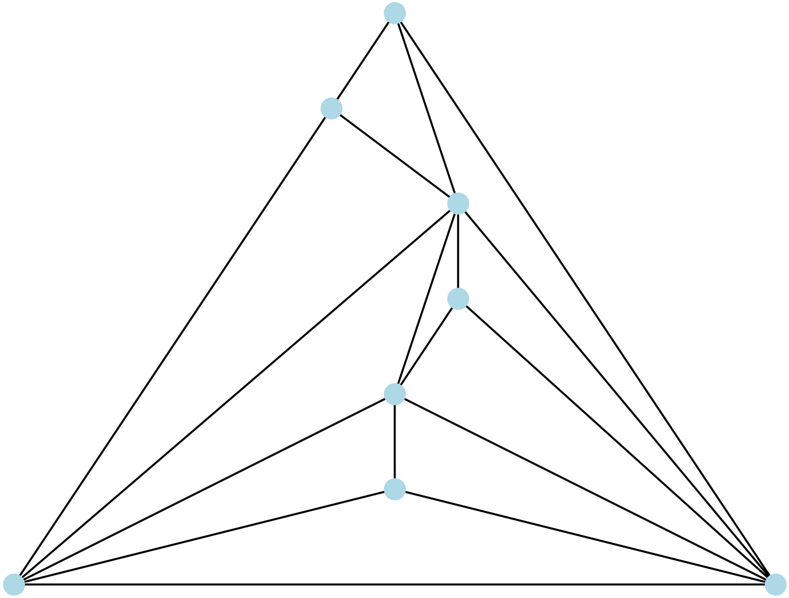
Lemma

Let G be a 3-connected plane graph. Then the set of boundaries of its faces is exactly the set of its non-separating induced cycles.

Proof.

On the whiteboard. □

Picture!



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Definition

Let G and G' be two plane graphs, and let $\varphi : V(G) \to V(G')$ be a bijection satisfying the following conditions:

1. $xy \in E(G) \iff \varphi(x)\varphi(y) \in E(G')$;
2. $U \subseteq V(G)$ is the set of boundary vertices of some face of G if and only if $\varphi(U) = \{\varphi(x) : x \in U\}$ is the set of boundary vertices of some face of G'.

Then φ is called an *isomorphism of plane graphs* G and G', and the plane graphs themselves are called *isomorphic*.

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Are all drawings of a planar graph are isomorphic?

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Then φ is called an *isomorphism of plane graphs* G and G', and the plane graphs themselves are called *isomorphic*.

Theorem (H. Whitney, 1933)

Any two plane drawings of a 3-connected graph G are isomorphic as plane graphs.

Proof.

On the whiteboard. □