Lecture 11, Planar Graphs+

19.12.2024

Content

Theorem (L. Euler, 1752)

Let G be a plane graph with v vertices, e edges, and f faces, having k connected components. Then

$$v - e + f = 1 + k.$$

Corollary

Let G be a planar graph without loops or multiple edges, and $v \geq 3$. Then the following statements hold:

1. $e \leq 3v - 6$.
2. *If G is bipartite, then $e \leq 2v - 4$.*

Corollary

If G is a planar graph without loops or multiple edges, then $\delta(G) \leq 5$.

Corollary

The graphs K_5 and $K_{3,3}$ are non-planar.

Definition

A graph H' is called a *subdivision* of a graph H if H' can be obtained from H by replacing some edges with simple paths (each replaced edge xy is replaced by a simple xy-path). All added vertices are distinct and have degree 2.
Vertices of H' corresponding to the vertices of H are called *principal vertices*.

- The notation $G \supset H$ means that G contains a subgraph that is a subdivision of H.

Corollary

1. *A subdivision of a graph H is planar if and only if H is planar.*
2. *Any subdivision of K_5 or $K_{3,3}$ is non-planar.*

Definition

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Formulate in terms of paths, what does it mean that $G \supset K_5$?

Lemma

Let $x, y \in V(G)$, $xy \in E(G)$. Then the following assertions hold:

1) *If $G \cdot xy \supseteq K_{3,3}$, then $G \supseteq K_{3,3}$.*
2) *If $G \cdot xy \supseteq K_5$, then $G \supseteq K_5$ or $G \supseteq K_{3,3}$.*

Proof. *Let $w = x \cdot y$, and let H be the subgraph $G \cdot xy$, which is a subdivision of $K_{3,3}$ or K_5.*

- *If $w \notin V(H)$, then clearly $G \supseteq K_{3,3}$ or $G \supseteq K_5$, respectively.*
- *Next, suppose $w \in V(H)$. Construct a subgraph H' of G as follows:*

$$V(H') = V(H) \setminus \{w\} \cup \{x, y\}.$$

  Include in $E(H')$ all edges of $E(H)$ not incident to w. For every edge $aw \in E(H)$, include in $E(H')$ one of the edges ax or ay (if both exist in G, choose any one of them). Finally, include the edge xy in $E(H')$.
- *Denote the edges of $H' \setminus xy$ incident to x as red, and the edges of $H' \setminus xy$ incident to y as blue. Together, the red and blue edges equal $d_H(w)$.*
- *If H' contains no blue edges, then $H' - y$ is a subgraph of G isomorphic to H. The same applies to the red edges. In this case, the proof of the lemma is complete.*

Continuation of the Proof of Lemma 8

- *Suppose* ay *is the only blue edge in* H'. *Then the edge* $aw \in E(H)$ *corresponds to the path* ayx *in* H', *i.e.,* H' *is a subdivision of* H *(Picture!).*

Continuation of the Proof of Lemma 8

- *Suppose ay is the only blue edge in H'. Then the edge $aw \in E(H)$ corresponds to the path ayx in H', i.e., H' is a subdivision of H (Picture!). In this case, the lemma is proven, and the argument is analogous for the case where there is exactly one red edge.*
- *Now suppose there are at least two red edges and two blue edges. Then $d_H(w) \geq 4$, from which it immediately follows that $H \supseteq K_5$ and $d_H(w) = 4$.*
- *Let z_1, z_2, z_3, z_4 be the four remaining primary vertices of the graph H. Each pair of vertices w, z_1, z_2, z_3, z_4 is connected in H by a path—a subdivision of the corresponding edge of K_5. These paths have no common internal vertices. These paths correspond to paths in H'.*
- *In H', there exist the paths xz_1, xz_2, yz_3, and yz_4 (Picture!). Thus, $H' \supseteq K_{3,3}$: each vertex of x, z_3, z_4 is connected by a path to each vertex of y, z_1, z_2, and these paths have no common internal vertices.*

□

Theorem (K. Kuratowski, 1930)

A graph G (possibly with multiple edges and loops) is non-planar if and only if G contains a subgraph that is a subdivision of K_5 or $K_{3,3}$.

Proof. $\Leftarrow$ (why?)

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A graph G (possibly with multiple edges and loops) is non-planar if and only if G contains a subgraph that is a subdivision of K_5 or $K_{3,3}$.

Proof. $\Leftarrow$ Trivially

$\Rightarrow$ *Assume the contrary and consider a minimal counterexample G (a non-planar graph that does not contain subdivisions of K_5 or $K_{3,3}$). Any graph that does not contain subdivisions of K_5 or $K_{3,3}$ and has fewer vertices than G, or the same number of vertices but fewer edges, must necessarily be planar.*

Statement 1

The graph G does not have loops or multiple edges.

Proof.

(why?)

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Statement 1

The graph G does not have loops or multiple edges.

Proof.

Let e be a loop in G. Then G is planar, and from its planarity it follows that the graph $G - e$ is also planar (a loop can always be redrawn in a planar embedding of $G - e$). Same for multiple edges. □

Statement 2

The graph G is 3-connected.

Proof.

- *If G is disconnected, then?*

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Proof.

- *If G is disconnected, then its components do not contain subdivisions of K_5 and $K_{3,3}$, and therefore, are planar. This implies G is planar, which is a contradiction.*
- *Suppose G has a cut-vertex a. then?*

Statement 2

The graph G is 3-connected.

Proof.

- *If G is disconnected, then its components do not contain subdivisions of K_5 and $K_{3,3}$, and therefore, are planar. This implies G is planar, which is a contradiction.*
- *Suppose G has a cut-vertex a. Then $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2) = \{a\}$.*
- *The graphs G_1 and G_2 do not contain subdivisions of K_5 or $K_{3,3}$, and thus, are planar.*
- *This implies that G is planar as well (it is possible to embed G_1 and G_2 such that a lies on the boundary of the outer face in both embeddings, and then glue them together). Contradiction.*

Continuation of Statement 2

- *Finally, suppose* G *is 2-connected but has a separating set* $S = \{a, b\}$*. Then* $G = G_1 \cup G_2$*, where* $V(G_1) \cap V(G_2) = S$*. then?*

Continuation of Statement 2

- *Finally, suppose* G *is 2-connected but has a separating set* $S = \{a, b\}$*. Then* $G = G_1 \cup G_2$*, where* $V(G_1) \cap V(G_2) = S$*. Let* $G'_i = G_i + ab$*.*
- *Assume* G'_1 *contains a subgraph* H *that is a subdivision of* K_5 *or* $K_{3,3}$*. Since* H *cannot be a subgraph of* G*, the edge* $ab \in E(H) \setminus E(G)$*.*
- *However,* G *contains an ab-path* P *passing through the vertices of* G_2*. Replacing the edge ab in* H *with the path* P*, we obtain a subdivision* H' *of the graph* H*, which is a subgraph of* G*. Thus,* G *contains a subdivision of* K_5 *or* $K_{3,3}$*, which is a contradiction.*
- *Therefore,* G'_1 *does not contain subdivisions of* K_5 *or* $K_{3,3}$*, and thus,* G'_1 *is planar. Similarly,* G'_2 *is planar.*
- *Hence, it is possible to embed these graphs in the plane such that the edge ab lies on the boundary of the outer face in both embeddings, and then glue these embeddings together. Contradiction.*

□

Proof of the Theorem (Continuation)

- *Clearly,* $G \neq K_4$*. Then, there exists an edge* $xy \in E(G)$ *such that the graph* $G \cdot xy$ *is 3-connected. Let* $w = x \cdot y$*.*
- *Then, we have* $G \cdot xy \not\supseteq K_5$*, and* $G \cdot xy \not\supseteq K_{3,3}$*, so the graph* $G \cdot xy$ *is planar.*
- *Let* $G' = G \cdot xy - w \cong G - x - y$*.*
- *Consider a planar embedding of the graph* G'*, obtained from the embedding of* $G \cdot xy$ *by removing the vertex* w*. Let* q *be the face of* G' *on which the vertex* w *is located.*
- *The graph* G' *is 2-connected, so the boundary of the face* q *is a simple cycle* Z*.*
- *Mark the vertices of* Z *adjacent to* y *(denote their set as* A*) and number them cyclically as* $a_1, a_2, \ldots, a_n$*. From the 3-connectedness of* G*, it follows that* $n \geq 2$*. Let* B *be the set of vertices of the cycle* Z *adjacent to* x*.*
- *If* $A = B$ *and* $n \geq 3$ *(since the graph* $G - A$ *is disconnected in this case), then?*

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- *If* $A = B$ *and* $n \geq 3$ *(since the graph* $G - A$ *is disconnected in this case), then* G *contains a subdivision of* K_5 *with primary vertices* $x, y, a_1, a_2, \dots, a_n$*, which is a contradiction.*
- *Next, assume* $B \subsetneq A$*. Let* $b \in B \setminus A$ *lie on the arc* $L = a_1 Z a_2$*, which contains no other vertices from* A*.*

Proof of the Theorem (Continuation) II

- *Suppose that there is a vertex* $b' \in B$ *that does not lie on the arc* L *(possibly coinciding with one of the vertices in* A*), but* $b' \notin \{a_1, a_2\}$.
- Then the cyclic order of the vertices a_1, b, a_2, b' on Z is as shown. then?

Proof of the Theorem (Continuation) II

- *Suppose that there is a vertex $b' \in B$ that does not lie on the arc L (possibly coinciding with one of the vertices in A), but $b' \notin \{a_1, a_2\}$.*
- Then the cyclic order of the vertices a_1, b, a_2, b' on Z is as shown. Therefore, G contains a subdivision of $K_{3,3}$ with main vertices x, a_1, a_2 (one partition) and y, b, b' (second partition), leading to a contradiction (see Fig. a).
- The remaining case is when all vertices of the set B lie on the arc L (possibly coinciding with a_1 or a_2).
- In this case, consider the original planar embedding of the graph $G \cdot xy$ and remove all edges from w to the vertices in $B \setminus A$.
- The edges from A to w divide the face q into n regions, one of which is the face d, bounded by L and the edges wa_1, wa_2.
- We can place the vertex x inside d and connect it with edges to w and the vertices from B, without violating planarity. To construct a planar embedding of G, it remains to rename w as y.

□

Content

Definition

1) A face is called a *triangle* if its boundary contains exactly three vertices.
2) A planar graph without loops is called a *triangulation* if every face is a triangle. Multiple edges are allowed.
3) *Triangulating* a planar graph means adding additional edges to it to achieve a triangulation.

Draw a triangulation of some graph with multiple edges

Definition

1) A face is called a *triangle* if its boundary contains exactly three vertices.
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3) *Triangulating* a planar graph means adding additional edges to it to achieve a triangulation.

Corollary

A triangulation is a 2-connected graph.

Lemma

Let G be a planar graph without loops, $v(G) \geq 3$ and $|V_d| \geq 3$ for any face d. Then G can be triangulated without introducing new pairs of multiple edges.

Proof:

- Let G not be a triangulation. Then G has a face d that is not a triangle. Let $H = G[V_d]$.
- Any two vertices in V_d can be connected by a polygon line in d that does not intersect the edges of the graph G.
- Thus, if the graph H is incomplete, we can add an edge without introducing new pairs of multiple edges.
- end of the proof?

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- Any two vertices in V_d can be connected by a polygon line in d that does not intersect the edges of the graph G.
- Thus, if the graph H is incomplete, we can add an edge without introducing new pairs of multiple edges.
- Let $H = K_m$. Since $|V_d| \geq 3$ and the graph H is planar, $m \in \{3, 4\}$.
- Since $B(d)$ is not a triangle, it is a cycle of length 4. Then two diagonals of this cycle must lie outside the boundary of f, which is clearly impossible: such diagonals would intersect each other.

Statement

Triangulation is a maximal planar graph.

Proof.

Let $2n$ be the number of faces in some triangulation T. Then, T has $3n$ edges. From Euler's formula we know that $v(T) = 1 + e(T) - f(T) = n + 2$. Hence, $e(T) = 3v(T) - 6$. □

Lemma

In any triangulation $T \subseteq G$, $v(T) \geq 4$, *there exists an edge* e *belonging to exactly two triangles — the two faces sharing* e.

take any edge?

Lemma

In any triangulation $T \subseteq G$, $v(T) \geq 4$, there exists an edge e belonging to exactly two triangles — the two faces sharing e.

Proof:

- We call a *separating* triangle — one that has vertices on both sides of the plane.
- If T has no separating triangles, the statement is obvious — any edge will suffice.
- Suppose there are separating triangles. Consider a separating triangle abc such that no other separating triangles are contained within it.
- However, there are vertices inside abc, which implies the presence of some edge e. Then e cannot belong to the separating triangle, as such a triangle would have to be contained within abc, contradicting the choice of abc.

□

Theorem (K. Wagner, 1936)

Let G be a planar graph without multiple edges. Then there exists a planar drawing of G in which all edges are straight-line segments.

Proof:

- We will prove the statement by induction on the number of vertices in the graph. The base case, for a graph with ≤ 3 vertices, is obvious.
- It is sufficient to prove the theorem for the case where G is a triangulation, any graph can be triangulated without introducing multiple edges. If we straighten the triangulation, the original graph will also be straightened.
- Take an edge $e = uv \in E(G)$ such that it belongs to exactly two triangles — the faces xuv and yuv.
- Let $G' = G \cdot uv$ be the triangulation obtained by "contracting" the faces xuv and yuv, while keeping the other faces of G unchanged. There are no multiple edges in G'.
- By the induction hypothesis, there exists a planar drawing of G' in which all edges are straight-line segments. We now consider this drawing further.

Continuation of the Proof:

- Order the vertices in $N_G(u)$ in the clockwise order of edges leaving u: $x, v, y, a_1, \ldots, a_k$. Since G is a triangulation, any two consecutive vertices in this order, together with u, form a triangular face.
- Similarly, order the vertices in $N_G(v)$ in the clockwise order of edges leaving v: $y, u, x, b_1, \ldots, b_m$. Since G is a triangulation, any two consecutive vertices in this order, together with v, form a triangular face.
- Then, in the graph G', the vertices in $N_{G'}(w)$ (the neighbors of w after the contraction of uv) will be ordered clockwise in the following sequence: $y, a_1, \ldots, a_k, x, b_1, \ldots, b_m$. Any two consecutive vertices (with respect to the edges leaving w) together with w will form a triangular face.
- Draw G from G' by turning w into u, v.

Content

- It is clear that a triangulation is not a bipartite graph. What can we say about the possibility of coloring the vertices of a triangulation with three colors?
- It is easy to see that a triangulation with such a coloring cannot have vertices of odd degree. (why?)

- It is clear that a triangulation is not a bipartite graph. What can we say about the possibility of coloring the vertices of a triangulation with three colors?
- It is easy to see that a triangulation with such a coloring cannot have vertices of odd degree. But can we color the vertices of a triangulation in three colors if all its vertices have even degree?

Theorem (L. I. Golovina, I. M. Yaglom, 1961)

Let T be a planar triangulation. Then $\chi(T) = 3$ if and only if all vertices of T have even degrees.

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Proof

- For any vertex $a \in T$, the graph $T(N_T(a))$ (the subgraph induced by neighbors of a) contains a cycle of length $d_T(a)$ (since the endpoints of the edges leaving a are distinct and adjacent in the planar embedding). Denote this cycle as $Z_{T,a}$.

$\Rightarrow$ The necessity of the condition is obvious. Indeed, suppose a vertex a has an odd degree. Then the cycle $Z_{T,a}$ is odd, and a proper 3-coloring must use all three colors. However, in this case, it is impossible to color vertex a.

$\Leftarrow$ Suppose all vertices of T have even degrees. We will prove the existence of a proper 3-coloring of T by induction on $v(T)$. The base case for $v(T) = 3$ is obvious.

Inductive Step:

- Suppose $v(T) > 3$, and let $a \in V(T)$ be a vertex of minimum degree. By one of the previous Corollaries, $d_T(a) \leq 5$. Since this degree is even, $d_T(a) \in \{2, 4\}$.
- Case $d_T(a) = 2$ is straightforward: $N_T(a) = \{b_1, b_2\}$, and b_1, b_2 are connected by two multiple edges e and e'.

- It is easy to see that $T' = T - a - e'$ is smaller than T and is a triangulation whose vertices all have even degrees. Thus, $\chi(T') = 3$ by the induction hypothesis, and a can be easily recolored to complete the coloring of T.
- Let $d_T(a) = 4$, and $N_T(a) = \{b_1, b_2, b_3, b_4\}$, where these vertices are listed in the order of traversal around the cycle $Z = Z_{T,a}$.
- Then the boundary of one of the faces of the graph $H = T - a$ is the cycle Z (assume this face is internal), and the boundaries of all other faces are triangles.
- Define a pair of vertices b_i, b_{i+2} as *good* if $b_i \neq b_{i+2}$ and $b_i b_{i+2} \notin E(T)$; otherwise, the pair is *bad*.
- If the pair b_1, b_3 is bad, then either $b_1 = b_3$ or the edge $b_1 b_3$ lies in the external region of the cycle Z. Clearly, in this case, the pair b_2, b_4 is good.
- Now assume the pair b_2, b_4 is good. Then the graph $T' = H \# b_2 b_4$ is a triangulation smaller than T, where $b = b_2 \# b_4$.
- The degrees of the vertices in the triangulation T' remain even:

$$d_{T'}(b_1) = d_T(b_1) - 2,$$
$$d_{T'}(b_3) = d_T(b_3) - 2,$$
$$d_{T'}(b) = d_T(b_2) + d_T(b_4) - 4.$$

- By the induction hypothesis, the triangulation T' admits a proper 3-coloring ρ' with $\rho'(b) = 1$.
- Construct a proper 3-coloring ρ of the vertices of T. For all vertices $v \in V(T) \setminus \{a, b_2, b_4\}$, set $\rho(v) = \rho'(v)$. Let $\rho(b_2) = \rho(b_4) = \rho'(b)$.
- Suppose $Z_{T,b_2} = ab_1x_1 \dots x_tb_3$ and $Z_{T,b_4} = ab_3y_1 \dots y_sb_1$. Clearly, both s and t are odd.
- Then the cycle $Z' = Z_{T,b} = b_1x_1 \dots x_tb_3y_1 \dots y_sb_1$ is colored in ρ' using colors 2 and 3. Therefore, the vertices b_1 and b_3, which are at even distances from each other, must have the same color. Thus, $\rho'(b_1) = \rho'(b_3) = 2$.
- Consequently, the vertices of the cycle $Z_{T,a}$ are colored in ρ with colors 1 and 2. Assigning $\rho(a) = 3$ completes the proper 3-coloring of the triangulation T.

□