Lecture 12, Constructible Graphs, Hypergraph Coloring

13.01.2025

Content

Definition (Hajós construction, 1961)

Let $q \in \mathbb{N}$. The class of q-constructible graphs $\mathcal{C}_q$ consists of graphs that can be obtained from K_q by any sequence of the following two operations:

- If $G \in \mathcal{C}_q$, $x, y \in V(G)$, $xy \notin E(G)$, then $G\#xy \in \mathcal{C}_q$.
- Let $G_1, G_2 \in \mathcal{C}_q$, $V(G_1) \cap V(G_2) = \{x\}$, $xy_1 \in E(G_1)$, $xy_2 \in E(G_2)$. Then

$$(G_1 - xy_1) \cup (G_2 - xy_2) + y_1y_2 \in \mathcal{C}_q.$$

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Lemma (Hajós, 1961)

Let $q \in \mathbb{N}$ and a graph G contains subgraph from $\mathcal{C}_q$, then $\chi(G) \geq q$.

Proof.

On the whiteboard. □

Definition (Ore, 1967)

1. Define the operation of *graph merging*. Let G_1, G_2 be graphs with $V(G_1) \cap V(G_2) = \varnothing$, and let $W_1 \subset V(G_1)$, $W_2 \subset V(G_2)$ be such that $|W_1| = |W_2|$ and $\mu : W_1 \to W_2$ is a bijection. Suppose $x_1y_1 \in E(G_1)$, $x_2y_2 \in E(G_2)$, where $x_1 \in W_1$, $\mu(x_1) = x_2$, and $\mu(y_1) \neq y_2$ (possibly, $y_1 \notin W_1$).
The *merging* of graphs G_1 and G_2 is the graph $G_1 \#_{\mu, x_1y_1, x_2y_2} G_2$, obtained from
$$(G_1 - x_1y_1) \cup (G_2 - x_2y_2) + y_1y_2$$
by merging the pairs of vertices $v, \mu(v)$ for all $v \in W_1$.
2. Let $q \in \mathbb{N}$. Define $\mathcal{C}'_q$ as the class of all graphs that can be obtained from K_q by any sequence of graph merging operations.

Lemma (Ore, 1967)

For any $q \geq 3$, $\mathcal{C}'_q \subset \mathcal{C}_q$.

Lemma (A. Urquhart, 1997.)

Let $q \in \mathbb{N}$. Then any graph G with $\chi(G) \geq q$ can be obtained using graph merging operations from graphs containing cliques of size at least q.

Content

What is a proper coloring of a hypergraph?

Definition

A coloring of the vertices of a hypergraph $\mathcal{H}$ is called *proper* if every hyperedge contains at least two vertices of different colors.

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An *image* of a hypergraph $\mathcal{H}$ is any graph G (possibly with multiple edges) such that $V(G) = V(\mathcal{H})$ and there exists a bijection $\varphi : E(G) \to E(\mathcal{H})$ such that $e \subseteq \varphi(e)$ for every edge $e \in E(G)$. We call φ the bijection of the image G.

- Multiple edges of the graph-image G, corresponding to different hyperedges of the hypergraph $\mathcal{H}$, are considered distinct.
- What is the connection between proper colorings of hypergraph and images?

From now on, every hyperedge $\mathcal{H}$ contains at least r vertices.

Definition

Let $r \geq 3$, and let G be an image of the hypergraph $\mathcal{H}$. Consider a sequence of vertices $a_0 b_0 a_1 b_1 \ldots a_n$ of the hypergraph $\mathcal{H}$ satisfying the following conditions:

(1) For each $i \in [0, n-1]$, the vertices a_i, b_i, and a_{i+1} are distinct, and there exists a hyperedge $e_i \in E(\mathcal{H})$ such that $a_i, b_i, a_{i+1} \in e_i$.

(2) All hyperedges $e_0, \ldots, e_{n-1}$ are distinct, and $a_0 b_0, \ldots, a_{n-1} b_{n-1} \in E(G)$, with $\varphi(a_i b_i) = e_i$.

Then $a_0 b_0 a_1 b_1 \ldots a_n$ is called an *alternating chain* from a_0 to a_n. The number n is called the *length* of this alternating chain. We say that the alternating chain passes through the vertices $a_0, b_0, \ldots, a_n$ and the edges $a_0 b_0, \ldots, a_{n-1} b_{n-1}$.

Picture!

- We allow the case $n = 0$ in the definition of an alternating chain. Thus, the vertex a_0 is considered an alternating chain from a_0 to a_0 of length 0.
- Since φ is a bijection, all edges $a_0b_0, \dots, a_{n-1}b_{n-1}$ in the definition of the alternating chain are distinct. Recall that multiple edges of the graph G, corresponding to different hyperedges of the hypergraph $\mathcal{H}$, are considered distinct.
- The vertices in the definition of an alternating chain are not required to be distinct. It is possible for an alternating chain to pass through some vertices more than once.

Lemma

Let $\mathcal{H}$ be a hypergraph, where every hyperedge contains at least r vertices, with $\Delta(\mathcal{H}) = \Delta$ and $k = \left\lceil \frac{2\Delta}{r} \right\rceil$. Then there exists an image G of the hypergraph $\mathcal{H}$ such that $\Delta(G) \leq k$.

Proof.

- For $r = 2$, for any image G of the hypergraph $\mathcal{H}$, we have $\Delta(G) \leq \Delta = k$. From now on, assume $r \geq 3$.
- For the graph G, let $V_{k+1}(G)$ denote the set of all vertices in G with degree at least $k+1$, and let $s_{k+1}(G)$ denote the sum of the degrees of the vertices in $V_{k+1}(G)$ in the graph G.
- Assume, for the sake of contradiction, that the lemma is false. In this case, for any image G, we have $V_{k+1}(G) \neq \varnothing$ and $s_{k+1}(G) > 0$.
- Choose the image G that minimizes $s_{k+1}(G)$. Denote the bijection of the image G by φ, and let $S = V_{k+1}(G)$.

Let U be the set consisting of all vertices of the graph G that are endpoints of alternating chains starting at the vertices in S, and let $F = G(U)$. Clearly, $S \subseteq U$.

Lemma (Property 1)

For any edge $e \in E(F)$, it holds that $\varphi(e) \subseteq U$.

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Lemma (Property 1)

For any edge $e \in E(F)$, it holds that $\varphi(e) \subseteq U$.

Lemma (Property 2)

If $u \in U$, $v \notin U$, and $uv \in E(G)$, then all vertices of the hyperedge $\varphi(uv)$ except v lie in U.

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For any vertex $u \in U$ it holds that $\deg_G(u) \geq k$.

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For any vertex $u \in U$ it holds that $\deg_G(u) \geq k$.

Let $u_1, \ldots, u_\ell$ be all the vertices in the set U that have degree less than k in F. Define:

$$t_i = d_G(u_i) - d_F(u_i), \quad t = \sum_{i=1}^{\ell} t_i.$$

Theorem (H.V. Gravin, D.V. Karpov, 2011)

Let $\mathcal{H}$ be a hypergraph where each hyperedge contains at least r vertices, $\Delta(\mathcal{H}) = \Delta$, and $k = \lceil \frac{2\Delta}{r} \rceil$.

1. *The vertices of $\mathcal{H}$ can be properly colored with $k+1$ colors.*
2. *If $r \geq 3$ and $k \geq 3$, then the vertices of $\mathcal{H}$ can be properly colored with k colors.*