Lecture 2, Matchings

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Let $k, g, n \in \mathbb{N}$ be such that $k, g \geq 3$ and $n > k^{g+1}$, s.t. $kn \equiv_2 0$. Then there exists G such that $g(G) = G$, $v(G) = n$ and $\delta(G) = \Delta(G) = k$.

Sequence $d_1, \ldots d_n$ is *graphical* if there is a graph G with such degree sequence.

Havel–Hakimi, 1955, 1962

Sequence $(s, t_1, \ldots, t_s, d_1, \ldots, d_n)$ is graphical iff $(t_1 - 1, \ldots, t_s - 1, d_1, \ldots, d_n)$ is graphical.

A set $U \subseteq V(G)$ is *independent*, iff $\forall u, v \in U \implies uv \notin E(G)$. The size of the maximum independent set is called $\alpha(G)$.
A set $M \subseteq E(G)$ is a *matching* iff and two edges from M do not share the same vertex. The size of the maximum matching is called $\alpha'(G)$. Matching is full if it touches all vertices of G.
A set $W \subseteq V(G)$ is a *vertex cover*, if it covers all edges, the minimum denoted as $\beta(G)$.
A set $F \subseteq E(G)$ is a *edge cover* if it covers all vertices, the minimum denoted as $\beta'(G)$.

Lemma

1. $U \subseteq V(G)$ is an independent set iff $V(G) \setminus U$ is a vertex cover.
2. $\alpha(G) + \beta(G) = v(G)$.

Theorem (T. Gallai, 1959)

Let G *s.t.* $\delta(G) > 0$, *then* $\alpha'(G) + \beta'(G) = v(G)$.

Let M be a matching in a graph G.

1. A path is called M-*alternating* if its edges alternate between edges in M and edges not in M.
2. An M-alternating path is called M-*augmenting* if its endpoints are not covered by the matching M.

Theorem (C. Berge, 1957)

A matching M in a graph G is maximum if and only if there are no M-augmenting paths.

Let $G = (V_1, V_2, E)$ be a bipartite graph with parts V_1 and V_2.

Theorem (P. Hall, 1935)

A bipartite graph G has a matching that covers all vertices of V_1 if and only if for any subset $U \subset V_1$, the following holds:

$$|U| \leq |N_G(U)|.$$

The condition on the size of the neighborhood from Hall's theorem will be called *Hall's condition* for part V_1.

Corollary

If $\delta(V_1) \geq k$ and $\Delta(V_2) \leq k$, then there is a matching covering V_1.

For an arbitrary graph G, let $o(G)$ denote the number of odd components of G (that is, the number of connected components containing an odd number of vertices).

Theorem (W. T. Tutte, 1947)

A graph G has a perfect matching if and only if for any $S \subset V(G)$ the following condition holds: $o(G - S) \leq |S|$.

A graph in which all vertices have degree 3 is called *cubic*.
A *bridge* of a graph is an edge that does not belong to any cycle.
If $S \subseteq V(G)$ s.t. $o(G - S) > |S|$, then we say that S is a *Tutte's set* of the graph G.

Theorem (Petersen, 1891)

Let G be a connected cubic graph with at most two bridges. Then G has a perfect matching.

The *edge-connectivity* $\lambda(G)$ is the size of a smallest edge cut.

Theorem (Plesnik, 1972)

Let G be a regular with degree k and $v(G) \equiv_2 0$, s.t. $\lambda(G) \geq k-1$. Let G' be a graph obtained from G by removing at most $k-1$ edges. Then, there is a perfect matching in G'.

Corollary

Let G be a regular degree k graph with $v(G) \equiv_2 0$. Also, $\lambda(G) \geq k-1$, then for each edge $e \in E(G)$ there is a perfect matching containing e.

A *k-factor* of a graph G is a spanning k-regular subgraph.

- A perfect matching is a 1-factor.

Theorem (J. Petersen, 1891)

Every $2k$-regular graph has a 2-factor.

Corollary

1. *A* $2k$*-regular graph is the union of* k *of its 2-factors.*
2. *For any* $r \leq k$*, a* $2k$*-regular graph has a* $2r$*-factor.*

Theorem (C. Thomassen, 1981)

Let G be a graph such that $\delta(G) \geq k$ and $\Delta(G) \leq k+1$. Let $r < k$, then there is a spanning subgraph H of G such that $\delta(H) \geq r$ and $\Delta(H) \leq r+1$.

Theorem (L. Lovasz, 1970)

Let $s, t \in \mathbb{N}$*, then any graph* G *s.t.* $\Delta(G) \leq s + t - 1$*, can be split into two graphs* H_1, H_2 *s.t.* $G = H_1 \cup H_2$ *and* $\Delta(H_1) \leq s$*,* $\Delta(H_2) \leq t$*.*

Definition

We define $\operatorname{def}(G) = v(G) - 2\alpha'(G)$, i.e.

$$\alpha'(G) = \frac{v(G) - \operatorname{def}(G)}{2}$$

Theorem (C. Berge, 1958)

For any graph G the following holds:

$$\operatorname{def}(G) = \max_{S \subseteq V(G)} \left(o(G - S) - |S|\right).$$

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