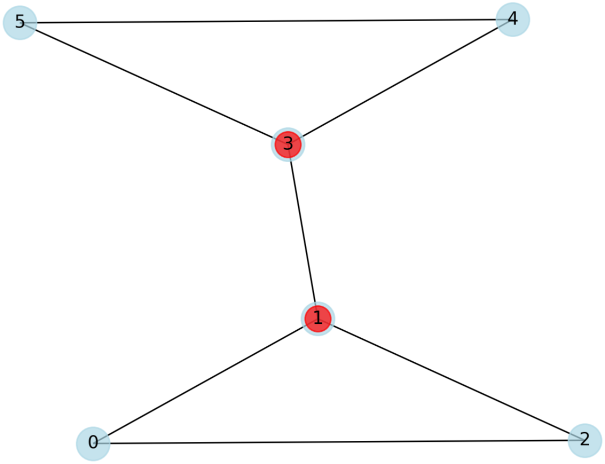
Lecture 3, Introduction to Connectivity

24.10.2024

- A vertex $a \in V(G)$ is called an *articulation point* if the graph $G - a$ is disconnected.
- A *block* is any maximal connected subgraph of G that does not contain articulation points.
- Due to maximality, a block of graph G is an induced subgraph of G on its vertex set.
- Any subgraph H of graph G without articulation points is included in at least one block (since H can be extended to a maximal subgraph without articulation points).


5
4
3
1
0
2

Lemma

Let B_1 *and* B_2 *be two different blocks of the graph* G*, with* $V(B_1) \cap V(B_2) \neq \varnothing$*. Then* $V(B_1) \cap V(B_2)$ *consists of an articulation point* a *of the graph* G*, where* a *is the only articulation point separating* B_1 *from* B_2*.*

Definition

Let $B(G)$ be bipartite graph, where the vertices of one part are the articulation points $a_1, \ldots, a_n$ of the graph G, and the vertices of the other part are its blocks $B_1, \ldots, B_m$. The vertices a_i and B_j are adjacent if $a_i \in V(B_j)$. There are no other edges in this graph. The graph $B(G)$ is called the *block and articulation point tree* of the graph G.

Lemma

Let B_1 and B_2 be two different blocks of the graph G, and let P be a path between them in the graph $B(G)$. Then the articulation points of the graph G that separate B_1 from B_2 are exactly those articulation points that lie on the path P. Ot her articulation points do not even separate the union of the blocks along the path P.

Theorem

1. *The block and articulation point tree is indeed a tree, with all its leaves corresponding to blocks.*
2. *An articulation point a separates two blocks B_1 and B_2 in the graph G if and only if a separates B_1 and B_2 in $B(G)$.*

Definition

We call a block B *extreme* if it corresponds to a leaf of the block and articulation point tree. The *interior* $\mathrm{Int}(B)$ of a block B is the set of all its vertices that are not articulation points in the graph G.

Theorem

Let B be an extreme block of a connected, but not biconnected graph G with $v(G) \geq 2$, and let $G' = G - Int(B)$. Then the graph G' is connected, and the blocks of G' are all the blocks of G except for B.

Let $U_1, \ldots, U_k$ be all the connected components of the graph $G - a$, and let $G_i = G(U_i \cup \{a\})$. We decompose the graph G into the graphs $G_1, \ldots, G_k$.

Lemma

Let $b \in U_i$. Then b separates the vertices $x, y \in V(G_i)$ in G_i if and only if b separates them in G. All the articulation points of the graphs $G_1, \ldots, G_k$ are exactly all the articulation points of the graph G except a.

Algorithm for Constructing the Block and Articulation Point Tree

- Choose an articulation point a and split G at a — replacing the graph G with the resulting graphs $G_1, \ldots, G_k$.
- In each of the graphs $G_1, \ldots, G_k$, construct the block and articulation point trees. Let $B(G_i) = T_i$.
- In the graph G_i, the vertex a is not an articulation point.
- Take the unique block B_i in G_i that contains a.
- Construct the tree $B(G)$ by joining the trees $T_1, \ldots, T_k$ at the point a (attaching T_i by the edge aB_i).

Let $X, Y \subseteq V(G)$, $R \subseteq V(G) \cup E(G)$.

Definition

We call the set R *separating* if the graph $G - R$ is disconnected. Let $\mathfrak{R}(G)$ denote the set of all separating sets of the graph G.

Definition

A graph G is *k-connected* if $v(G) \geq k + 1$ and the minimum vertex separating set in the graph G contains at least k vertices.

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