Lecture 4, k-Connected Graphs

31.10.2024

Let $X, Y \subseteq V(G)$, $R \subseteq V(G) \cup E(G)$.

Definition

We call the set R *separating* if the graph $G - R$ is disconnected. Let $\mathfrak{R}(G)$ denote the set of all separating sets of the graph G.

Definition

A graph G is *k-connected* if $v(G) \geq k + 1$ and the minimum vertex separating set in the graph G contains at least k vertices.

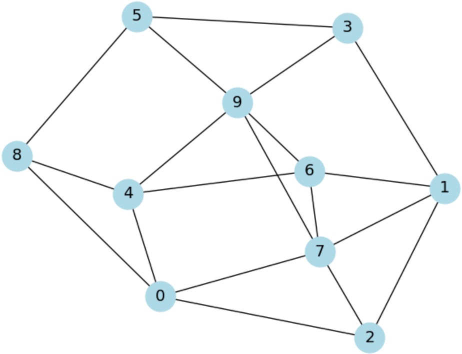
Definition

Let $X \not\subseteq R$, $Y \not\subseteq R$. We say that R *separates* the sets X and Y (or, equivalently, *separates* X and Y from each other) if no two vertices $v_x \in X$ and $v_y \in Y$ lie in the same connected component of the graph $G - R$.

1. Let $x, y \in V(G)$ be non-adjacent vertices. Denote by $\kappa_G(x, y)$ the size of the smallest set $R \subset V(G)$ such that R separates x and y. If x and y are adjacent, then we set $\kappa_G(x, y) = +\infty$. We call $\kappa_G(x, y)$ the *connectivity* of vertices x and y.
2. Let $X, Y \subset V(G)$. Denote by $\kappa_G(X, Y)$ the size of the smallest set $R \subset V(G)$ such that R separates X and Y. If no such set exists, we set $\kappa_G(X, Y) = +\infty$.

Theorem (Menger, 1927, Goring 2000)

[t] Let $X, Y \subset V(G)$, $\infty > \kappa_G(X, Y) \geq k$, $|X| \geq k$, $|Y| \geq k$. *Then in the graph* G, *there exist* k *disjoint* XY*-paths.*



Corollary

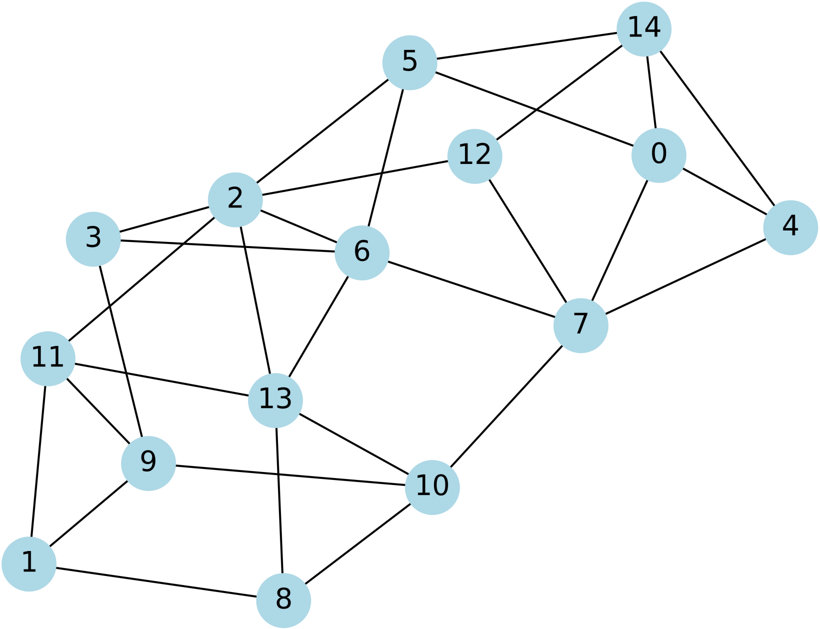
Let vertices $x, y \in V(G)$ *be non-adjacent,* $\kappa_G(x, y) \geq k$*. Then there exist* k *independent paths from* x *to* y*.*

Theorem (Whitney, 1932)

Let G be a k-connected graph. Then for any two vertices $x, y \in V(G)$, there exist k independent paths from x to y.

Let $\mathfrak{S} \subset \mathfrak{R}(G)$.

1. A set $A \subset V(G)$ is a *part of the* $\mathfrak{S}$*-partition* if no set from $\mathfrak{S}$ separates any two vertices from A, but any other vertex of the graph G is separated from A by at least one set from $\mathfrak{S}$.
   The set of all parts of the partition of graph G by the separating sets $\mathfrak{S}$ will be denoted as $\mathrm{Part}(\mathfrak{S})$. When it is unclear which graph is being partitioned, we will write $\mathrm{Part}(G; \mathfrak{S})$.
2. A vertex of a part $A \in \mathrm{Part}(\mathfrak{S})$ is called *internal* if it does not belong to any set from $\mathfrak{S}$. The set of such vertices will be called the *interior* of part A and denoted as $\mathrm{Int}(A)$. Vertices that belong to any set from $\mathfrak{S}$ are called *boundary vertices*, and their set — the *boundary* — is denoted by $\mathrm{Bound}(A)$.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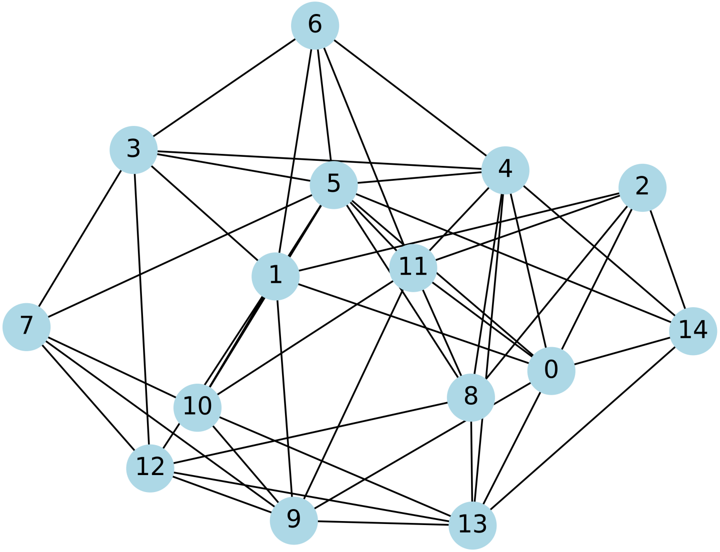

Figure: 4-connected

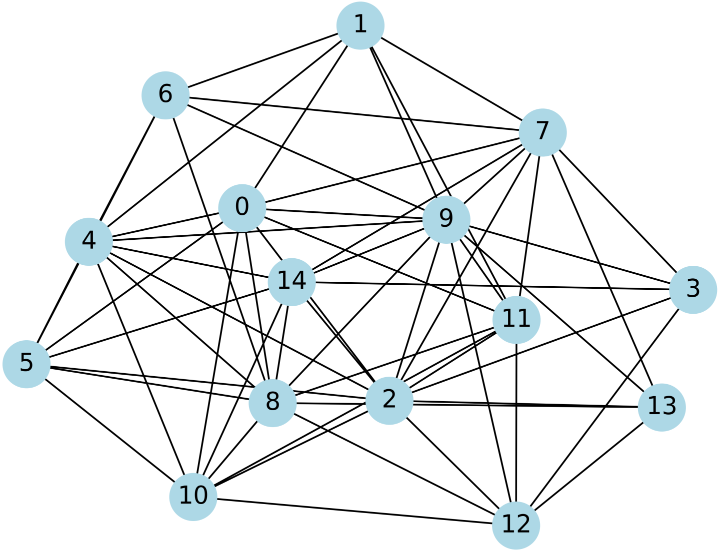

Figure: 5-connected

We denote by $\mathfrak{R}_k(G)$ the set of all k-vertex separating sets of the graph G.

Lemma

Let $\mathfrak{S} \subset \mathfrak{R}_k(G)$, $A \in \mathrm{Part}(\mathfrak{S})$. *Then the following statements hold.*

1. *A vertex* $x \in \mathrm{Int}(A)$ *is not adjacent to any vertices in the set* $V(G) \setminus A$.
2. *If* $\mathrm{Int}(A) \neq \varnothing$, *then* $\mathrm{Bound}(A)$ *separates* $\mathrm{Int}(A)$ *from* $V(G) \setminus A$.

Lemma

Let G be a k-connected graph, and let $\mathfrak{S}, \mathfrak{T} \subset \mathfrak{R}_k(G)$.

1. *Let $A \in \mathrm{Part}(\mathfrak{S})$. Then $\mathrm{Bound}(A)$ is the set of all vertices in part A that are adjacent to at least one vertex in $V(G) \setminus A$.*
2. *Let $A \in \mathrm{Part}(\mathfrak{S})$ and $A \in \mathrm{Part}(\mathfrak{T})$. Then the boundary of A as part of $\mathrm{Part}(\mathfrak{S})$ coincides with the boundary of A as part of $\mathrm{Part}(\mathfrak{T})$.*

Theorem

Let $\mathfrak{S}_1, \ldots, \mathfrak{S}_n \subset \mathfrak{R}(G)$, *and let* $\mathfrak{S} = \bigcup_{i=1}^n \mathfrak{S}_i$. *Consider all sets of vertices of the form*

$$A = \bigcap_{i=1}^{n} A_i, \quad \textit{where } A_i \in \mathrm{Part}(\mathfrak{S}_i). \tag{1}$$

Then the following statements hold:

1. *Any part* $A \in \mathrm{Part}(\mathfrak{S})$ *can be represented in the form (1).*
2. $A \in \mathrm{Part}(\mathfrak{S})$ *if and only if* A *is the maximal subset of vertices of the graph* G *representable in the form (1).*
3. *If a set of vertices* A *can be represented in the form (1) and* $A \notin \mathrm{Part}(\mathfrak{S})$, *then* A *is a subset of one of the sets in* $\mathfrak{S}$.

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