Lecture 5, Connectivity

07.11.2024

Let $X, Y \subseteq V(G)$, $R \subseteq V(G) \cup E(G)$.

Definition

We call the set R *separating* if the graph $G - R$ is disconnected. Let $\mathfrak{R}(G)$ denote the set of all separating sets of the graph G.

Definition

A graph G is *k-connected* if $v(G) \geq k + 1$ and the minimum vertex separating set in the graph G contains at least k vertices.

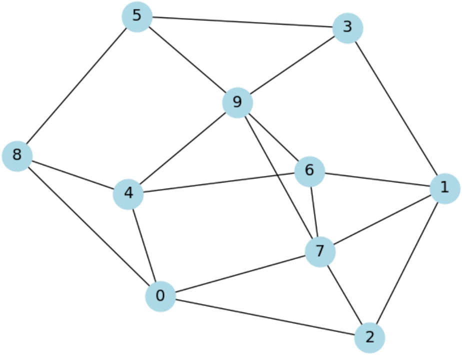
Definition

Let $X \not\subseteq R$, $Y \not\subseteq R$. We say that R *separates* the sets X and Y (or, equivalently, *separates* X and Y from each other) if no two vertices $v_x \in X$ and $v_y \in Y$ lie in the same connected component of the graph $G - R$.

1. Let $x, y \in V(G)$ be non-adjacent vertices. Denote by $\kappa_G(x, y)$ the size of the smallest set $R \subset V(G)$ such that R separates x and y. If x and y are adjacent, then we set $\kappa_G(x, y) = +\infty$. We call $\kappa_G(x, y)$ the *connectivity* of vertices x and y.
2. Let $X, Y \subset V(G)$. Denote by $\kappa_G(X, Y)$ the size of the smallest set $R \subset V(G)$ such that R separates X and Y. If no such set exists, we set $\kappa_G(X, Y) = +\infty$.

Theorem (Menger, 1927, Goring 2000)

[t] Let $X, Y \subset V(G)$, $\infty > \kappa_G(X,Y) \geq k$, $|X| \geq k$, $|Y| \geq k$. *Then in the graph* G, *there exist* k *disjoint* XY*-paths.*



Corollary

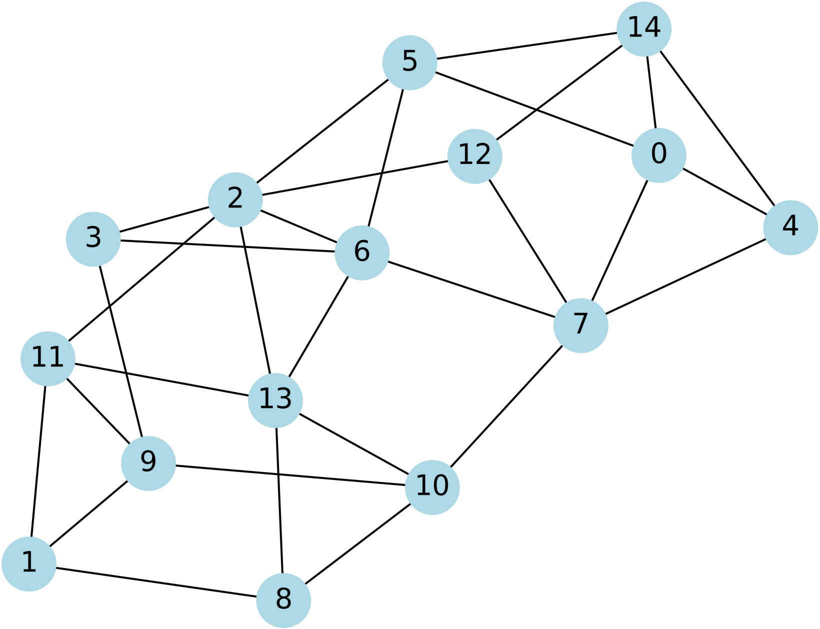
Let vertices $x, y \in V(G)$ *be non-adjacent,* $\kappa_G(x, y) \geq k$*. Then there exist* k *independent paths from* x *to* y*.*

Theorem (Whitney, 1932)

Let G be a k-connected graph. Then for any two vertices $x, y \in V(G)$, there exist k independent paths from x to y.

Let $\mathfrak{S} \subset \mathfrak{R}(G)$.

1. A set $A \subset V(G)$ is a *part of the* $\mathfrak{S}$*-partition* if no set from $\mathfrak{S}$ separates any two vertices from A, but any other vertex of the graph G is separated from A by at least one set from $\mathfrak{S}$.
   The set of all parts of the partition of graph G by the separating sets $\mathfrak{S}$ will be denoted as $\mathrm{Part}(\mathfrak{S})$. When it is unclear which graph is being partitioned, we will write $\mathrm{Part}(G; \mathfrak{S})$.
2. A vertex of a part $A \in \mathrm{Part}(\mathfrak{S})$ is called *internal* if it does not belong to any set from $\mathfrak{S}$. The set of such vertices will be called the *interior* of part A and denoted as $\mathrm{Int}(A)$. Vertices that belong to any set from $\mathfrak{S}$ are called *boundary vertices*, and their set — the *boundary* — is denoted by $\mathrm{Bound}(A)$.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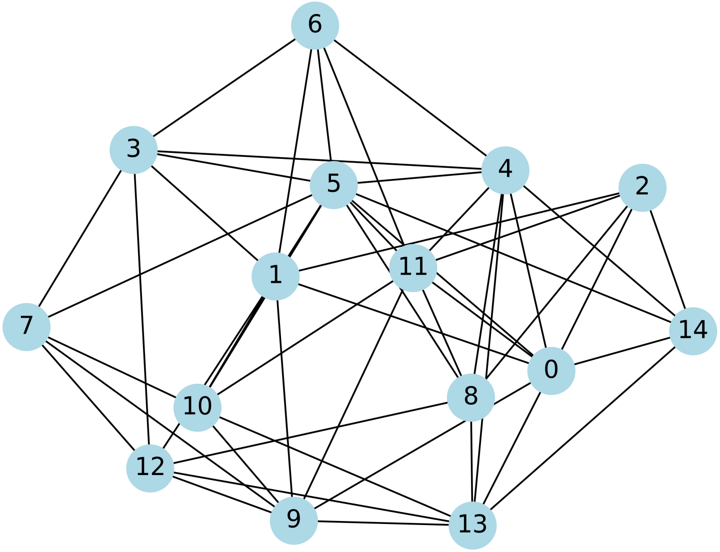


Figure: 4-connected

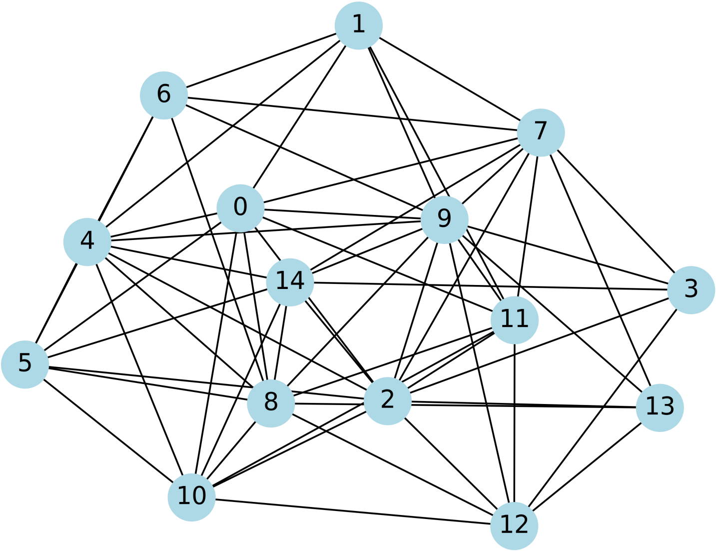


Figure: 5-connected

We denote by $\mathfrak{R}_k(G)$ the set of all k-vertex separating sets of the graph G.

Lemma

Let $\mathfrak{S} \subset \mathfrak{R}_k(G)$, $A \in \mathrm{Part}(\mathfrak{S})$. *Then the following statements hold.*

1. *A vertex* $x \in \mathrm{Int}(A)$ *is not adjacent to any vertices in the set* $V(G) \setminus A$.
2. *If* $\mathrm{Int}(A) \neq \varnothing$, *then* $\mathrm{Bound}(A)$ *separates* $\mathrm{Int}(A)$ *from* $V(G) \setminus A$.

Lemma

Let G be a k-connected graph, and let $\mathfrak{S}, \mathfrak{T} \subset \mathfrak{R}_k(G)$.

1. *Let* $A \in \mathrm{Part}(\mathfrak{S})$. *Then* $\mathrm{Bound}(A)$ *is the set of all vertices in part A that are adjacent to at least one vertex in* $V(G) \setminus A$.
2. *Let* $A \in \mathrm{Part}(\mathfrak{S})$ *and* $A \in \mathrm{Part}(\mathfrak{T})$. *Then the boundary of A as part of* $\mathrm{Part}(\mathfrak{S})$ *coincides with the boundary of A as part of* $\mathrm{Part}(\mathfrak{T})$.

Theorem

Let $\mathfrak{S}_1, \ldots, \mathfrak{S}_n \subset \mathfrak{R}(G)$, *and let* $\mathfrak{S} = \bigcup_{i=1}^n \mathfrak{S}_i$. *Consider all sets of vertices of the form*

$$A = \bigcap_{i=1}^{n} A_i, \quad \text{where } A_i \in \mathrm{Part}(\mathfrak{S}_i). \tag{1}$$

Then the following statements hold:

1. *Any part* $A \in \mathrm{Part}(\mathfrak{S})$ *can be represented in the form (1).*
2. $A \in \mathrm{Part}(\mathfrak{S})$ *if and only if* A *is the maximal subset of vertices of the graph* G *representable in the form (1).*
3. *If a set of vertices* A *can be represented in the form (1) and* $A \notin \mathrm{Part}(\mathfrak{S})$, *then* A *is a subset of one of the sets in* $\mathfrak{S}$.

Proof.

On the whiteboard.

Lemma

Let $\mathfrak{S}, \mathfrak{T} \subset \mathfrak{R}(G)$, *and let a part* $A \in \operatorname{Part}(\mathfrak{S})$ *be such that none of the sets in* $\mathfrak{T}$ *separate it. Then* $A \in \operatorname{Part}(\mathfrak{S} \cup \mathfrak{T})$.

Lemma

Let $\mathfrak{S}, \mathfrak{T} \subset \mathfrak{R}(G)$*, and let a part* $A \in \mathrm{Part}(\mathfrak{S})$ *be such that none of the sets in* $\mathfrak{T}$ *separate it. Then* $A \in \mathrm{Part}(\mathfrak{S} \cup \mathfrak{T})$*.*

Proof.

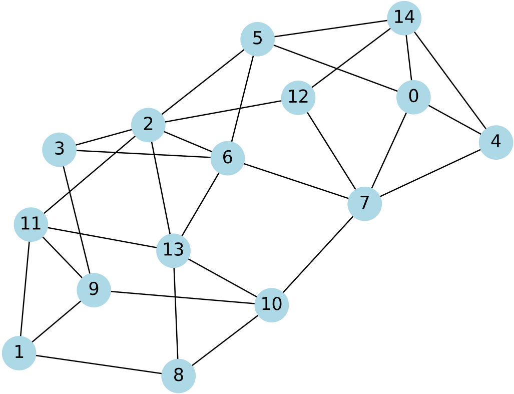
- None of the sets in $\mathfrak{S} \cup \mathfrak{T}$ separates A, so there exists a part $B \in \mathrm{Part}(\mathfrak{S} \cup \mathfrak{T})$ such that $A \subset B$.
- There exists a part $A' \in \mathrm{Part}(\mathfrak{S})$ containing B. Then $A \subset B \subset A'$, from which it is obvious that $A = B = A'$.

□

- **From now on, let G be a k-connected graph.**

Definition

We call distinct sets $S, T \in \mathfrak{R}_k(G)$ *independent* if S does not separate T and T does not separate S. Otherwise, we will call these sets *dependent*.



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Definition

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Lemma

Let $S, T \in \mathfrak{R}_k(G)$ and $A \in \mathrm{Part}(S)\colon T \cap \mathrm{Int}(A) = \varnothing$. Then T does not separate part A and, consequently, T does not separate set S.

Can it be that $\mathrm{Int}(A) = \varnothing$, for some $A \in \mathrm{Part}(S)$?

- **From now on, let G be a k-connected graph.**

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Proof.

- $G(\mathrm{Int}(A))$ is connected, and $\forall x \in S \setminus T$ is adjacent to at least one vertex in the set $\mathrm{Int}(A)$.
- Consequently, the graph $G(\mathrm{Int}(A) \cup (S \setminus T))$ is connected, from which it is evident that T does not separate A. $\square$

Lemma

Let $S, T \in \mathfrak{R}_k(G)$ *be such that the set* S *does not separate* T*. Then* T *and* S *are independent.*

Lemma

Let $S, T \in \mathfrak{R}_k(G)$ be such that the set S does not separate T. Then T and S are independent.

Proof.

- T can intersect the interior of at most one part of $\text{Part}(S)$. (why?)
- $\exists A \in \text{Part}(S)\colon\ \text{Int}(A) \cap T = \varnothing \implies T$ does not separate S.

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Lemma

Let $S, T \in \mathfrak{R}_k(G)$ be such that the set S does not separate T. Then T and S are independent.

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□

We conclude that one of two cases is possible: either the sets S and T separate each other (then they are *dependent*), or the sets S and T do not separate each other (then they are *independent*).

Lemma

Let $S, T \in \mathfrak{R}_k(G)$ *be independent, and* $A \in \operatorname{Part}(S)$ *contain* T*. Then in* $\operatorname{Part}(T)$ *there* $\exists! B \in \operatorname{Part}(T)\colon B \supset \operatorname{Part}(S) \setminus A$ *and* $\operatorname{Part}(T) \setminus B \subset A$*.*

Lemma

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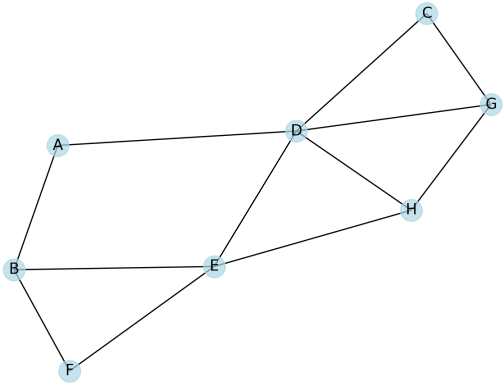
Proof.

- The set T does not intersect the interiors of parts of $\mathrm{Part}(S)$ distinct from A. Hence, the set T does not separate any part of $\mathrm{Part}(S)$ distinct from A.
- Since $S \setminus T \neq \varnothing$, all these parts are contained within a single part of $\mathrm{Part}(T)$. (why?)

□

Let the sets $S, T \in \mathfrak{R}_k(G)$ be *dependent*, with $\mathrm{Part}(S) = \{A_1, \ldots, A_m\}$, $\mathrm{Part}(T) = \{B_1, \ldots, B_n\}$, $P = T \cap S$, $T_i = T \cap \mathrm{Int}(A_i)$, $S_j = S \cap \mathrm{Int}(B_j)$, and $G_{i,j} = A_i \cap B_j$.



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What about separating set with more than 2 parts?

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Lemma

1. *All sets* $T_1, \ldots, T_m$*;* $S_1, \ldots, S_n$ *are non-empty.*
2. $\mathrm{Part}(\{S, T\}) = \{G_{i,j}\}_{i \in [1..m], j \in [1..n]}$, *with* $\mathrm{Bound}(G_{i,j}) = P \cup T_i \cup S_j$.

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Proof.

1. (why?) .

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Proof.

1. Trivially.
2. Parts $\mathrm{Part}(\{S, T\})$ are maximal by inclusion among sets of the form $G_{i,j}$. But $G_{\alpha,\beta} \not\subset G_{\gamma,\delta}$ for $(\alpha, \beta) \neq (\gamma, \delta)$.
   The statement $\mathrm{Bound}(G_{i,j}) = P \cup T_i \cup S_j$ is trivial.

□

$|\mathrm{Bound}(G_{i,j})| \geq k$ for any i, j since it is a separating set, right?

Lemma

1. *Let* $i \neq x$, $j \neq y$, $|\operatorname{Bound}(G_{i,j})| \geq k$ *and* $|\operatorname{Bound}(G_{x,y})| \geq k$. *Then* $|\operatorname{Bound}(G_{i,j})| = |\operatorname{Bound}(G_{x,y})| = k$, $|\operatorname{Part}(S)| = |\operatorname{Part}(T)| = 2$, $|T_i| = |S_y|$, *and* $|T_x| = |S_j|$.
2. *If all parts of* $\operatorname{Part}(\{S, T\})$ *contain at least* k *vertices, then each part of* $\operatorname{Part}(\{S, T\})$ *has exactly* k *vertices,* $|\operatorname{Part}(S)| = |\operatorname{Part}(T)| = 2$, *and* $|T_1| = |T_2| = |S_1| = |S_2|$.

Proof.

Whiteboard.

Definition

Let $S \in \mathfrak{R}_k(G)$, and let H be a connected component of the graph $G - S$. We will call H a *fragment*. We will call the set S the *boundary* of the fragment H and denote it by $\mathrm{Bound}(H)$.

Definition

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- Fragments are the interiors of parts of a partition of the graph G by a k-vertex separating set.
- We will show that the concepts of a fragment and its boundary have an independent meaning.

Lemma

Let H be a fragment in a k-connected graph G. Then $\mathrm{Bound}(H) = N_G(H)$.

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Proof.

Trivial. □

Can it happen that $\exists A \colon A \in \mathrm{Part}(S) \cap \mathrm{Part}(T)$ and $A \neq \varnothing$?

Lemma

Let H *be a fragment of the graph* G, $T \in \mathfrak{R}_k(G)$, *with* $T \cap H \neq \varnothing$, *and* T *is independent with* $\mathrm{Bound}(H)$. *Then* $T \not\supseteq H$ *and exists a fragment* $H' \subseteq H\colon\ \mathrm{Bound}(H') = T$.

Lemma

Let H be a fragment of the graph G, $T \in \mathfrak{R}_k(G)$, with $T \cap H \neq \varnothing$, and T is independent with $\mathrm{Bound}(H)$*. Then $T \not\supseteq H$ and exists a fragment $H' \subseteq H\colon$* $\mathrm{Bound}(H') = T$.

Proof.

- Let $S = \mathrm{Bound}(H)$, and let $A \in \mathrm{Part}(S)\colon H = \mathrm{Int}(A)$.
- $T \subset A$

Lemma

Let H be a fragment of the graph G, $T \in \mathfrak{R}_k(G)$, with $T \cap H \neq \varnothing$, and T is independent with $\mathrm{Bound}(H)$. Then $T \not\supseteq H$ and exists a fragment $H' \subseteq H\colon \mathrm{Bound}(H') = T$.

Proof.

- Let $S = \mathrm{Bound}(H)$, and let $A \in \mathrm{Part}(S)\colon H = \mathrm{Int}(A)$.
- $T \subset A$, since S, T are independent.
- Hence $\exists B \in \mathrm{Part}(T)$ s.t. $B \subset A$. Thus, $T \not\supseteq H$.

□

Bibliography I