Lecture 6. Connectivity and Colourings

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Abstract

Below you will see some problems, some of them just problems from your homework and some more related to our lecture topics. The entire set would be counted as a homework (so it is a chance to get some extra points). Please, indicate in our spreadsheet which tasks you have solved. You are allowed to work in groups and use lecture notes. If I refer to some theorem (without stating it), then it means that we've already proved it on lectures and you can find it in the slides. Also, some problems have **hints** right in your sheet, you should read them together with the statement. And there are several lemmas and theorems that you do **not** need to prove and can use freely. On the next lecture, we will discuss the problems and I will prove all the statements.

By $G \cdot e$, where $e \in E(G)$ we denote the contraction of the edge e in the graph G.

1. (hw1, 11) $\Delta(G) \leqslant 2000$ (maximum degree of G). Prove that we can color halves of edges (color for one side and color for the other side) in colors from the set $\{1, 2, 3, \ldots, 2000\}$ such that halves of a single edge are differ by exactly 1. And that all halves incident to any vertex are colored differently.

   Hint: Use some theory about factors of a graph. Complete to the regular graph, then use Petersen's two factor theorem. Color edges according to the factors.

2. (hw2, 5a) Let $n \geqslant 5$. The edges of the complete graph K_n are colored black and white. Prove that the vertices of K_n can be divided into two groups V_1 and V_2 such that there exists a Hamiltonian path in $G(V_1)$ consisting of white edges, and a Hamiltonian path in $G(V_2)$ consisting of black edges.

   Hint: Prove that there is a Hamiltonian path, such that at first it uses only white edges, and then only black edges by induction.

3. (hw4, 4) Let G be a biconnected graph with $v(G) \geqslant 4$ and $e \in E(G)$. Prove that at least one of the graphs $G - e$ and $G \cdot e$ is biconnected.

   Hint: Use blocks and articulation points tree. Let T be a blocks and articulation points tree of $G - e$ and $G - e$ is not biconnected. Let $ab = e$, then it is easy to see that a, b in two different blocks (and they are not articulation points) and that T is just a path. Hence, $G \cdot e$ is biconnected.

4. (hw4, 5) The graph G is connected. We call a *separator* a minimal set of vertices whose removal disconnects the graph. Let S and R be separators of the graph G. It is known that S does not separate R. Prove that R does not separate S (it is possible that $|R| \neq |S|$).

   Hint: Use reasoning similar to the proof from Lecture (but it was for k-connected graphs).

5. (hw4, 6) Prove that any graph with a minimum degree of $2k$ contains a $(k + 1)$-edge-connected subgraph. (A graph is called n-edge-connected if it remains connected when fewer than n edges are removed.)

   Hint: Prove by induction statement that is stronger that the problem statement.

   Hint2: Make statement stronger: if sum of degrees is at least $2v(G)k - 2k + 1$, then each cut has size at least k.

Definition 1. *By $\chi(G)$ we denote the chromatic number of a graph G. I.e. the minimal number of colours needed to colour the vertices of G such that no two adjacent vertices have the same colour.*

6. Draw a graph with $\Delta(G) \leqslant d$ such that $\chi(G) > d$.
7. Let G be a connected graph, $\Delta(G) \leqslant d$ and $\delta(G) \leqslant d-1$. Prove that $\chi(G) \leqslant d$.
8. Let $G\colon \Delta(G) = \delta(G) = d$ and G is connected, but not biconnected. Prove that $\chi(G) \leqslant d$.

Definition 2. *Let $k \in \mathbb{N}$. We say that a graph G is k-reducible, if its vertices can be enumerated $v_1, \dots, v_n$ such that each vertex v_i is disjoint with at most k vertices from the set $\{v_{i+1}, \dots, v_n\}$.*

9. Prove that if for any subgraph H of G we have $\delta(H) \leqslant k-1$, then G is k-reducible.
10. Prove that if G is k-reducible, then for any subgraph H of G we have $\delta(H) \leqslant k-1$.

Definition 3. *We say that G is k-critical, if $\chi(G) = k$ and for any subgraph $H \subsetneq G$ we have $\chi(H) < k$.*

11. Prove that if G is k-critical, then $\delta(G) \geqslant k-1$.
12. Let G – k-critical and $S \subset V(G)$ its separating set, $|S| < k$. Prove that $G[S]$ is not a complete graph. Where $G[S]$ denoted a subgraph of G induced by the set S.
13. Prove that $\forall k \in \mathbb{N}$ there is a graph G that is triangle-free, $\chi(G) = k$.

    Hint: You've seen it on Discrete Mathematics course.

Definition 4. *For any number $k \in \mathbb{N}$, denote $\chi_G(k)$ – the number of right colourings of the graph G into k colours.*

14. Let $e \in E(G)$, prove that $\chi_{G-e}(k) = \chi_G(k) + \chi_{G\cdot e}(k)$.
15. Prove that if G has no self-loops, then $\chi_G(k)$ is a polynomial of degree $\nu(G)$ over $\mathbb{Z}$. Moreover, if $n = \nu(G)$ and $\chi_G(k) = a_0 + a_1 k + \ldots + a_n k^n$, then $a_n \geqslant 0$, $a_{n-1} \leqslant 0$, $a_{n-2} \geqslant 0$, $a_{n-3} \leqslant 0$ and so on.
16. If $G_1, \dots, G_n$ are connected components of the graph G, then $\chi_G(k) = \prod_{i=1}^{n} \chi_{G_i}(k)$. And that if p is a 0 multiplicity (i.e. $\chi_G(k) = k^p \cdot h(k)$ and $h(0) \neq 0$), then the number of components is p.
17. (hw4, 7a) The edges of a complete graph on 4000 vertices vertices are colored in three colors. Prove that this graph contains a monochromatic simple cycle of odd length at least 41.

    Hint: Use the fact that any graph with average degree $x \geqslant 2$ contains a cycle of length at least x. Consider colour with maximal number of edges.