Lecture 8, Spanning Trees

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Content

Denote by $st(G)$ the number of spanning trees of a connected graph G.

Theorem (A. Cayley, 1889)

Let G be a graph where loops and multiple edges are allowed, and let $e \in E(G)$ be an edge that is not a loop. Then

$$st(G) = st(G - e) + st(G * e).$$

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$$st(G) = st(G - e) + st(G * e).$$

Proof.

- The number of spanning trees of the graph G that do not contain the edge e is obviously equal to $st(G - e)$.
- There is a bijection between the spanning trees containing the edge e and the spanning trees of the graph $G * e$, given by $T \to T * e$ (where T is a spanning tree of G, $e \in E(T)$).

Picture! □

Theorem (C. Cayley, 1889)

$st(K_n) = n^{n-2}$.

H. Prüfer, 1918.

On the whiteboard. □

Content

By $u(T)$ denote the number of leaves in a tree T.

Theorem (S. Schuster, 1983)

Let a connected graph G have spanning trees with m and n leaves, where $m < n$. Then, for any natural number $k \in [m, n]$, there exists a spanning tree of G with exactly k leaves.

Proof.

- Let T_1 and T^* be spanning trees with $u(T_1) = n$ and $u(T^*) = m$, respectively.
- Starting from the tree T_1, we will perform the following step iteratively. Assume that a sequence of spanning trees $T_1, \ldots, T_i$ of G has been constructed.
- If $T_i \neq T^*$, then there exists an edge $e_i \in E(T^*) \setminus E(T_i)$. Let $G_i = T_i + e_i$.
- If $T_i \neq T^*$, then there is an edge $e_i \in E(T^*) \setminus E(T_i)$. Let $G_i \coloneqq T_i + e_i$. Picture!
- In the graph G_i, there is exactly one simple cycle C_i that includes the edge e_i. Clearly, $E(C_i) \not\subseteq E(T^*)$, so there exists an edge $f_i \in E(C_i) \setminus E(T^*)$. Define

$$T_{i+1} = G_i - f_i = T_i + e_i - f_i.$$

- Since the tree T_{i+1} contains more edges from $E(T^*)$ than T_i, at some point, we must reach $T_\ell = T^*$.
- Consider the sequence of trees $T_1, T_2, \ldots, T_\ell = T^*$.
- The trees T_i and T_{i+1} differ by exactly two edges. Therefore,

$$|u(T_i) - u(T_{i+1})| \leq 2.$$

  Hence, the numbers of leaves in the trees of this sequence cover the interval $[m, n]$ without skipping more than one number.

- Let $t \in [m, n]$, and suppose there is no tree with t leaves in our sequence.
- Then there exists some j such that $u(T_j) = t + 1$ and $u(T_{j+1}) = t - 1$. By construction, $T_{j+1} = G_j - f_j$ and $T_j = G_j - e_j$, where $f_j = ab$ and $e_j = xy$. Picture!
- Then $d_{G_j}(a) = d_{G_j}(b) = 2$ and $d_{G_j}(x) > 2$, $d_{G_j}(y) > 2$. (why?)

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- Then $d_{G_j}(a) = d_{G_j}(b) = 2$ and $d_{G_j}(x) > 2$, $d_{G_j}(y) > 2$. Since both vertices a and b become leaves after removing the edge e_j, and vertices x and y do not become leaves after removing the edge f_j).
- Thus, in the cycle C_j, there are vertices of degree 2 and vertices of degree greater than 2. Hence, one of the edges $e' = uw \in E(C_j)$ has $d_{G_j}(u) > 2$ and $d_{G_j}(w) = 2$. Then, in the tree $T' = G_i - e'$, exactly one vertex in $V(C_i)$, namely w, becomes a leaf, so $u(T') = t$.

  Picture!

□

Content

1 Number of spanning trees

2 Intermediate Value Theorem

3 Number of leaves in a spanning tree of a connected graph with $\delta(G) \geq 3$

Theorem (D. J. Kleitman, D. B. West, 1991)

In a connected graph G with $\delta(G) \geq 3$, there exists a spanning tree with at least $\frac{v(G)}{4} + 2$ leaves.

Picture!

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Proof.

- We present an algorithm to construct a spanning tree with the desired number of leaves. The algorithm will iteratively identify a tree in G, step by step, by adding vertices.
- Assume that at some point, we have already constructed a tree F, which is a subgraph of G.

Definition

- A leaf x of the tree F is called *dead* if all vertices of G adjacent to x are included in the tree F.
- The number of dead vertices of the tree F is denoted by $b(F)$.

Dead vertices will remain dead leaves at all subsequent stages of the construction. For the tree F, we define

$$\alpha(F) = \frac{3}{4}u(F) + \frac{1}{4}b(F) - \frac{1}{4}v(F).$$

- We aim to construct a spanning tree T of the graph G such that $\alpha(T) \geq 2$. (why?)

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- We aim to construct a spanning tree T of the graph G such that $\alpha(T) \geq 2$.
- Since all leaves in the spanning tree are dead, it follows that

$$u(T) = b(T) = \frac{1}{4}v(G) + \alpha(T),$$

  and the tree T satisfies the required conditions.

Invariant.

- Suppose that after several steps of construction, we have obtained a tree F ($V(F) \subseteq V(G)$, $E(F) \subseteq E(G)$).
- Assume that as a result of the step, Δv vertices were added, the number of leaves increased by Δu, and the number of dead vertices increased by Δb.
- Define the *gain* of the step S as the quantity

$$P(S) = \frac{3}{4}\Delta u + \frac{1}{4}\Delta b - \frac{1}{4}\Delta v.$$

- We will perform only steps with non-negative gain. When calculating the gain of a step, we will assume that all added vertices that are not explicitly identified as dead are not dead. This assumption only reduces the gain of the step.
- Clearly, for the final spanning tree T, the value of $\alpha(T)$ will be the sum of $\alpha(F')$ (where F' is the base tree, whose construction will be described later) and the sum of the gains of all steps.
- We will describe several options for steps of the algorithm. We will proceed to the next option only when we confirm the impossibility of all previous options.

Steps of the Algorithm

Introduce the notation $W = V(G) \setminus V(F)$.

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$$P(S1) \geq \frac{3}{4} \cdot 1 - \frac{1}{4} \cdot 1 = \frac{1}{2}.$$

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S2. There is a vertex x in the tree F adjacent to at least two vertices in W. Add these two vertices to the tree, resulting in $\Delta v = 2$, $\Delta u = 1$. The gain is:

$$P(S2) \geq \frac{3}{4} \cdot 1 - 2 \cdot \frac{1}{4} = \frac{1}{4}.$$

S3. There is a vertex $y \in W$ adjacent to the tree F and at least two vertices in W. Add y and its two adjacent vertices to the tree, resulting in

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S3. There is a vertex $y \in W$ adjacent to the tree F and at least two vertices in W. Add y and its two adjacent vertices to the tree, resulting in $\Delta v = 3$, $\Delta u = 1$. The gain is:

$$P(S3) \geq \frac{3}{4} \cdot 1 - 3 \cdot \frac{1}{4} = 0.$$

Picture!

- S1: Each non-leaf in F is not adjacent to W.
- S2: Each leaf is adjacent to at most one vertex of W.
- S3: For any $y \in W$ if it is adjacent to F, then y has at most one neighbour in W.

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- If $F \neq \varnothing$, then there exists a vertex $y \in W$ adjacent to the tree F.
- $d_G(y) \geq 3$, so y is adjacent to two vertices $x, x' \in V(F)$. Connect y to x. Since it is impossible to perform $S1$ or $S2$, the vertex x' is a leaf in the tree F and is adjacent to exactly one vertex from W, which is y.
- Therefore, in the new tree,

- S1: Each non-leaf in F is not adjacent to W.
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- If $F \neq \varnothing$, then there exists a vertex $y \in W$ adjacent to the tree F.
- $d_G(y) \geq 3$, so y is adjacent to two vertices $x, x' \in V(F)$. Connect y to x. Since it is impossible to perform $S1$ or $S2$, the vertex x' is a leaf in the tree F and is adjacent to exactly one vertex from W, which is y.
- Therefore, in the new tree, x' is a dead vertex. Thus, $\Delta v = 1$, $\Delta b \geq 1$, and

$$P(S4) \geq \frac{1}{4} - \frac{1}{4} \geq 0.$$

Picture!

Construction of the Base Tree.

- We want to start with a base tree F' such that $\alpha(F') \geq \frac{3}{2}$. Then we will explain why, during the construction process, an additional $\frac{1}{2}$ will be added on top of the gains calculated for the steps. Let us consider two cases.

Case B1. The graph G has a vertex a of degree at least 4.

- The base tree F' is a tree where the vertex a is connected to $k \geq 4$ vertices from its neighborhood. We have:

$$v(F') = ?, \quad u(F') = ?,$$

  and

$$\alpha(F') \geq ?$$

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$$v(F') = k + 1, \quad u(F') = k,$$

and

$$\alpha(F') \geq \frac{3}{4} \cdot k - \frac{1}{4} \cdot (k+1) = \frac{2k-1}{4} > \frac{3}{2}.$$

Case B2. All vertices in G have degree 3.

- First, consider a tree F', where a vertex a is connected to three vertices b_1, b_2, b_3 from its neighborhood. Clearly,

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$$\alpha(F') \geq \frac{3}{4} \cdot 3 - \frac{1}{4} \cdot 4 = \frac{5}{4}.$$

- We are short by $\frac{1}{4}$, and if one of the three leaves of F' is dead, it contributes an additional $\frac{1}{4}$, making $\alpha(F') \geq \frac{3}{2}$.
- The remaining case is when each of the vertices b_1, b_2, b_3 is adjacent to at least one vertex outside $V(F')$.
- In our case, all vertices of G have degree 3, and the sum of the degrees of the vertices in the induced subgraph $G(\{a, b_1, b_2, b_3\})$ is even. Therefore, one of the vertices b_1, b_2, b_3 must be adjacent to at least two vertices outside $V(F')$.
- Then, we perform step $S2$ and add a gain of $\frac{1}{4}$, which suffices.

Picture!

To complete the proof of the theorem, it remains to show that the steps of the construction contribute an additional gain of at least $\frac{1}{2}$. To do this, let us analyze the end of the construction process.

- If the last step was $S1$, its gain is at least $\frac{1}{2}$.
- Suppose the last step was $S2$ or $S3$. In this case, we added two new leaves, which must be dead since no more steps can be performed.
- This contributes an additional gain of at least $\frac{1}{2}$. Note that even if the last step was $S2$ described in Case $B2$, the gain from these two dead vertices was not accounted for in that step.
- Suppose the last step was $S4$. Then, the added vertex y (see figure) turned out to be dead (contributing $\frac{1}{4}$), meaning that y was adjacent only to vertices in the tree F.
- However, $d_G(y) = 3$, meaning there were three such vertices, not just one, as accounted for in the description of step $S4$. Thus, we find two additional dead vertices, contributing $\frac{1}{2}$, thereby completing the proof of the theorem.

$\varnothing$

Definition. Let T be a spanning tree of a connected graph G.

- For any edge $e \in E(G) \setminus E(T)$, the graph $T + e$ contains a unique cycle C_e, which passes through e. We call C_e the *fundamental cycle* of the edge e with respect to the tree T.
- Let $f \in E(C_e), f \neq e$. Then $T' = T + e - f$ is also a spanning tree of G. We say that T' is obtained from T by *replacing* the edge e with f.

Definition.

- Consider ordered sets $\mathfrak{T} = (T_1, \ldots, T_k)$ of k spanning trees of the graph G. Let $\mathfrak{T}$ be the set of all such ordered sets.
- Define $E(\mathfrak{T}) = E(T_1) \cup \cdots \cup E(T_k)$ and $e(\mathfrak{T}) = |E(\mathfrak{T})|$.
- A sequence of edges $e_0, \ldots, e_n$ is called a *sequence of replacements* for $\mathfrak{T}$, starting with e_0, if:
  - $e_n \notin E(\mathfrak{T})$;
  - For every $i < n$, there exists an index $s(i)$ such that $e_i \in E(T_{s(i)})$, and e_{i+1} lies on the fundamental cycle of e_i with respect to the tree $T_{s(i)}$.