Lecture 9, Coloring

05.12.2024

Content

- A *coloring* of the vertices of a graph G into k colors is a function $\rho : V(G) \to M$, where $|M| = k$. The coloring ρ is called *proper* if $\rho(v) \neq \rho(u)$ for any pair of adjacent vertices u and v.
- The *chromatic number* of a graph G, denoted by $\chi(G)$, is the smallest natural number for which there exists a proper coloring of the vertices of G with that number of colors.

Lemma

Let G be a connected graph, $\Delta(G) \leq d$, and suppose that at least one vertex of G has degree less than d. Then $\chi(G) \leq d$.

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Picture!

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On the whiteboard.

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Theorem (R. L. Brooks, 1941)

Let $d \geq 3$, and let G be a connected graph, distinct from K_{d+1}, with $\Delta(G) \leq d$. Then $\chi(G) \leq d$.

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Definition

Choice number $\mathrm{ch}(G)$ is the smallest $k \in \mathbb{N}$. Such that if one assigns list of k colors to each vertex, then there is a proper coloring, such that color of each vertex belongs to the list of that vertex.

- In all cases where color lists are considered, we will denote the list with an uppercase letter (typically L), and its size with a lowercase letter: $\ell(v) = |L(v)|$.
- Clearly, $\mathrm{ch}(G) \geq \chi(G)$.

There exist graphs for which $\mathrm{ch}(G) > \chi(G)$.
Example?

Definition

A graph G is called *d-choosable* if for any set of lists L satisfying the condition $\ell(v) \geq d_G(v)$ for each vertex $v \in V(G)$, there exists a proper coloring of the vertices of G using colors from the lists.

- A list of colors satisfying the specified condition is called a *d-list*.

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Definition

A vertex $v \in V(G)$ is called *normal* if $\ell(v) = d_G(v)$, and *excessive* if $\ell(v) > d_G(v)$.

Lemma

Let G be a connected graph, and let L be a d-list in which a vertex a is excessive. Then there exists a proper coloring of the vertices of G in accordance with the list L.

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- A connected graph in which every block is either an odd cycle or a complete graph is called a *Gallai tree.*

Theorem (O. V. Borodin, 1977)

If a connected graph G is not a Gallai tree, then G is d-choosable.

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Theorem (V. G. Vizing, 1976)

Let $d \geq 3$, and let G be a connected graph, distinct from K_{d+1}, with $\Delta(G) \leq d$. Then $\mathrm{ch}(G) \leq d$.

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Definition

A graph G is called *k-critical* if $\chi(G) = k$, but $\chi(H) < k$ for every subgraph H of G.

Lemma

If G is a k-critical graph, then $\delta(G) \geq k - 1$.

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Let G be a k-critical graph, and let $S \subset V(G)$ be a separating set with $|S| < k$. Then the graph $G[S]$ is not complete.

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Theorem (T. Gallai, 1963)

Let $k \geq 3$, and let G be a k-critical graph. Let V_{k-1} be the set of all vertices of G with degree $k - 1$, and let $G_{k-1} = G[V_{k-1}]$. Then G_{k-1} is a Gallai forest.

Proof.

On the whiteboard.

Content

Definition

For any natural number k, let $\chi_G(k)$ denote the number of proper vertex colorings of a graph G using k colors. The function $\chi_G(k)$ is called the *chromatic polynomial* of the graph G.

- Thus, $\chi_G(\chi(G)) \neq 0$, and $\chi_G(k) = 0$ for any natural number $k < \chi(G)$.

Lemma

Let G be a non-empty graph, and let $e = uv$ be one of its edges. Then:

$$\chi_{G-e}(k) = \chi_G(k) + \chi_{G \cdot uv}(k),$$

where $G \cdot uv$ denotes the graph obtained by contracting the edge uv.

Theorem

For any loopless graph G, the following statements hold:

1. *The function $\chi_G(k) \in \mathbb{Z}[k]$ is a monic polynomial with integer coefficients of degree $|V(G)|$.*
2. *The signs of the coefficients of $\chi_G(k)$ alternate (i.e., the leading coefficient is non-negative, the next coefficient is non-positive, then non-negative again, and so on).*

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Lemma

Let $G_1, \ldots, G_n$ be the components of a graph G. Then: $\chi_G(k) = \prod_{i=1}^{n} \chi_{G_i}(k)$.

Theorem

For any graph G, the number 0 is a root of $\chi_G(k)$ with multiplicity equal to the number of connected components of G.

Proof.
On the whiteboard.

Lemma

Let G be a connected graph with n blocks $B_1, \ldots, B_n$. Then:

$$\chi_G(k) = \left(\frac{1}{k}\right)^{n-1} \cdot \prod_{i=1}^{n} \chi_{B_i}(k).$$

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Theorem (E. G. Whitehead, L.-C. Zhao, 1984)

Let G be a connected graph with more than one vertex. Then the number 1 is a root of the polynomial $\chi_G(k)$ with multiplicity equal to the number of blocks of the graph G.

Proof.

On the whiteboard.